An FPRAS for two-terminal reliability in directed acyclic graphs

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Joint work with Heng Guo (University of Edinburgh)

ICALP

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s-t network reliability

Two-terminal network reliability

Input: a graph $G = (V, E)$ and parameters $q_e \in (0,1)$

a source node s and a sink node t

Output: the *probability* that $s \to_{G(p)} t$ if each edge $e \in E$ fails independently with prob. q_e

s can reach t in the remaining graph

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Exact computing is #P-complete for ***directed*** / ***undirected*** / ***DAG*** / ***planar DAG*** graphs

Approximate s-t network reliability in DAGs

Two-terminal network reliability approximation

Input: a ***DAG (direct acyclic graph)*** $G = (V, E)$ and parameters $q_e \in (0,1)$

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an error bound $\varepsilon > 0$

Output: a ***random*** number $\hat{p}$ approximating s-t network reliability $p = \Pr[s \to_{G(p)} t]$

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Two-terminal network unreliability approximation

Output: a ***random*** number $\hat{q}$ approximating s-t network unreliability $1 - p$

$$\Pr[\hat{q} \in (1 \pm \varepsilon)(1 - p)] \geq \frac{2}{3}$$

Previous works

Direct Monte Carlo method (sample random subgraph and check reachability)

- the estimation is efficient if $p = \frac{1}{\mathrm{poly}(m)}$ (e.g. $q_e = O\left(\frac{\log m}{m}\right)$ is small, where $m = |E|$)

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Improved analysis on lower bound of s-t reliability [Zenklusen and Laumanns 10]

- Improve the lower bound of reliability for some special DAGs

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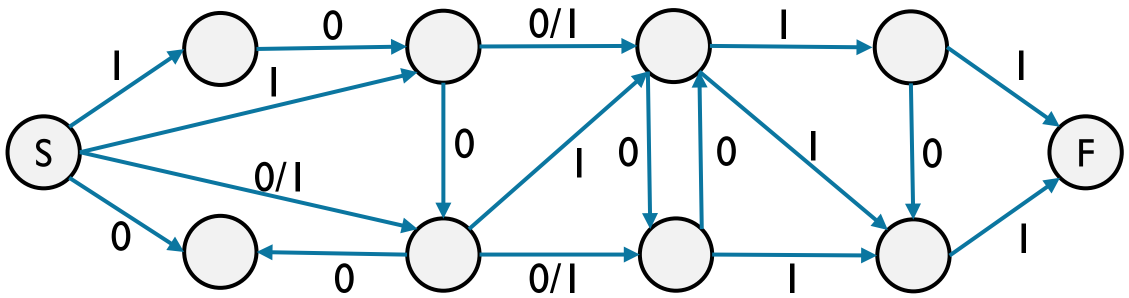
- Given n-state NFA, count #{distinct accepting strings} of length ℓ



An NFA

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Very recently, improved FPRAS for #NFA [Meel, Chakraborty and Mathur 24]

- An FPRAS for s-t reliability in time $\tilde{O}(m^{19})$ for $q_e = \frac{1}{2}$ via standard black-box reduction

Our results

There is an ***FPRAS*** for two-terminal network ***reliability*** in DAGs in time

$$\tilde{O}(n^6 m^4 \max\{m^4, \varepsilon^{-4}\}), \text{where } n = |V| \text{ and } m = |E|$$

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There is ***no FPRAS*** for two-terminal network ***unreliability*** in DAGs
unless there is an FPRAS for ***#BIS*** problem

#BIS: counting the number of independent sets in bipartite graphs
conjectured to has no FPRAS

Some basic settings for this talk

Two-terminal network reliability approximation

Input: a ***DAG (direct acyclic graph)*** $G = (V, E)$ and parameters $q_e \in (0,1)$

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Assumption

For any edge $e \in E, q_e = \frac{1}{2}$

general $q_e \in (0,1)$ can be solved with ***very small tweaks***

$$p = \frac{\#\{\text{subgraphs such that } s \text{ can reach } t\}}{2^m}$$

counting problem

Sampling and counting

Two-terminal network reliability in DAGs

$$\Omega = \{\text{subgraphs of } G \text{ such that } s \text{ can reach } t\}$$

- ***Counting Problem (network reliability)***: estimate the size $\#\Omega$
- ***Sampling Problem***: draw random subgraphs from Ω ***uniformly*** at random

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Approx. Counting $\Longleftrightarrow$ ***Sampling***

Self-Reducibility

[Jerrum, Valiant, Vazirani 86]

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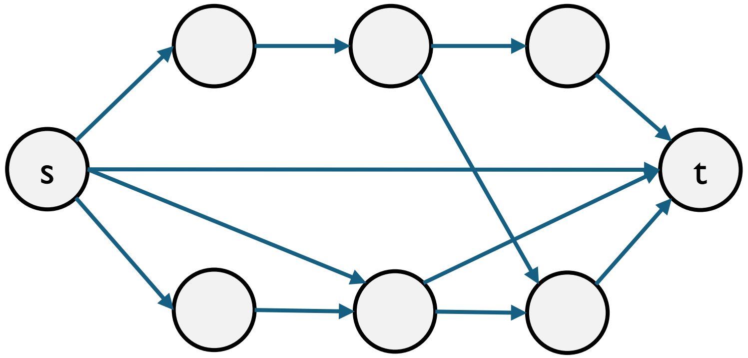
Approx. Counting $\xleftrightarrow{\text{Self-Reducibility [Jerrum, Valiant, Vazirani 86]}}$ ***Sampling***

Our algorithm

- Decompose the input two-terminal reliability instance into many ***sub-instances***
- Solve the ***sampling / counting*** problems recursively in sub-instances

Our Algorithm



sort all vertices according to topological order $s = v_n \succ v_{n-1} \succ \cdots \succ v_1 = t$

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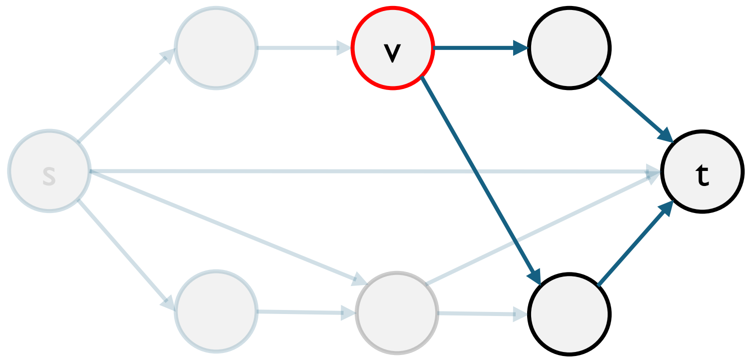

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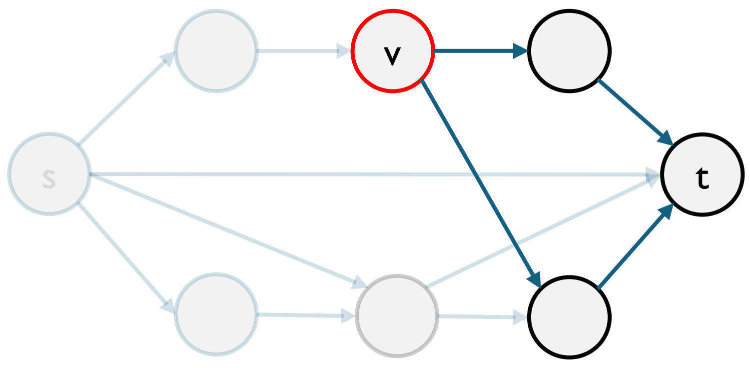

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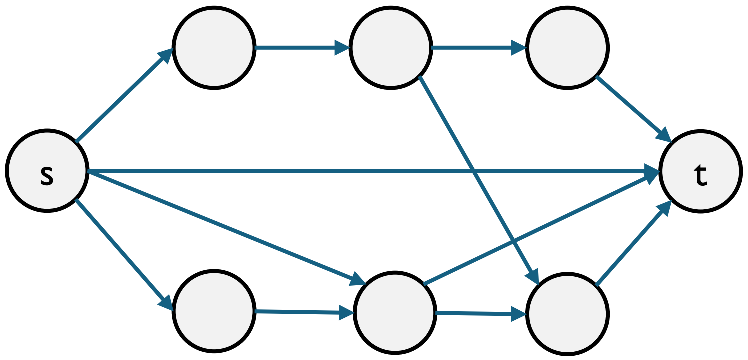

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Compute (Z_v, S_v) for v from v_1 to v_n via ***dynamic programming*** + ***Monte Carlo***

Framework appeared in [Gore, Jerrum, Kannan, Sweedyk, and Mahaney 97] [Arenas, Croquevielle, Jayaram and Riveros 21]

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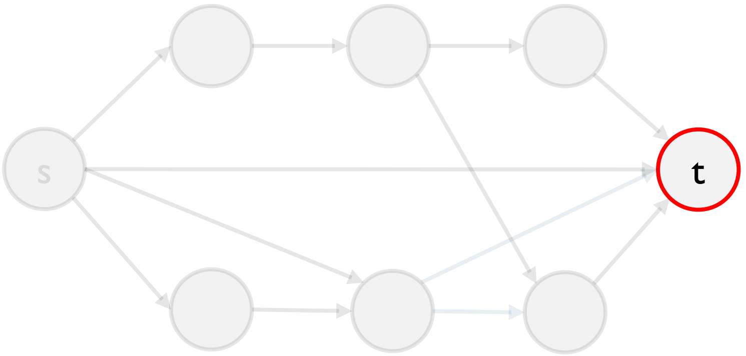

base case $v = t$ is trivial

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Estimate the count via Karp-Luby

- A vertex $v \in V$ and a set of ***neighbors*** $v_1, v_2, v_3, \ldots, v_d$

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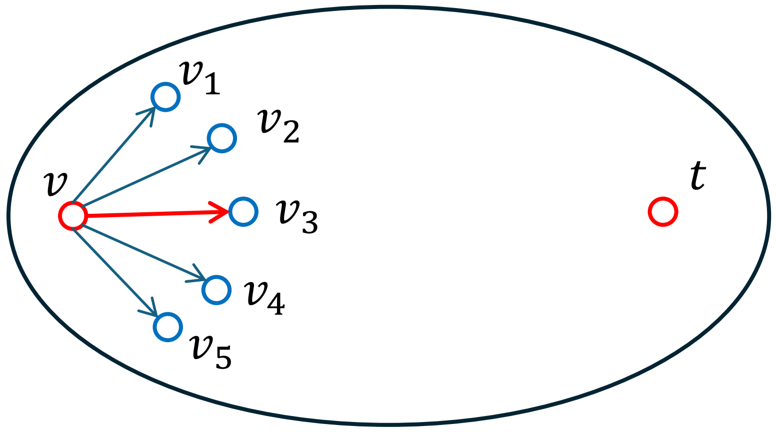

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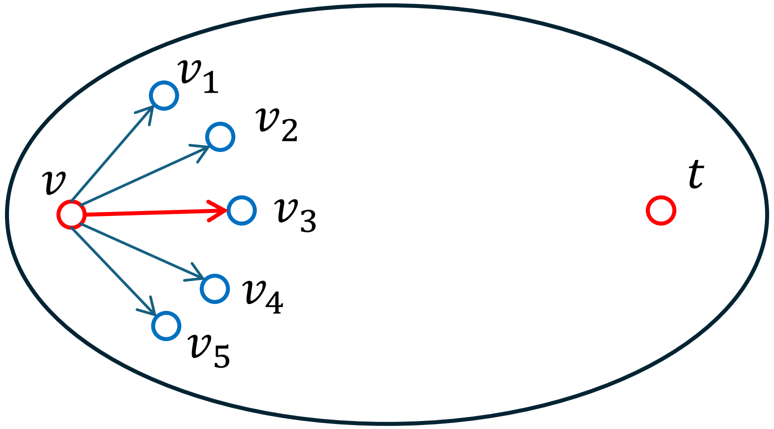

Karp-Luby's method for estimating the size of union [Karp and Luby 83]

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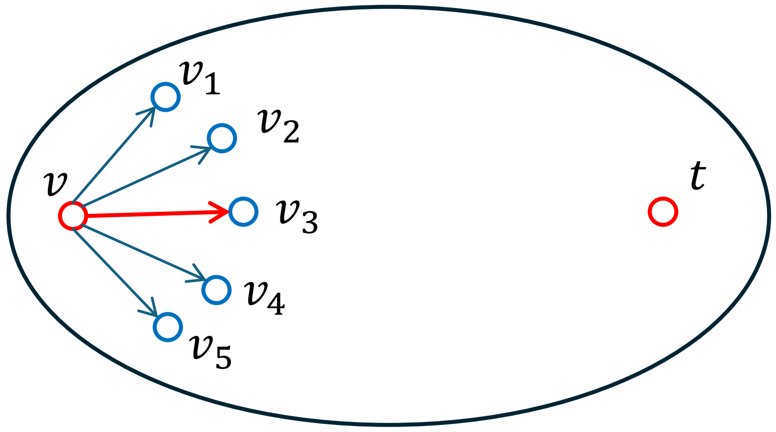

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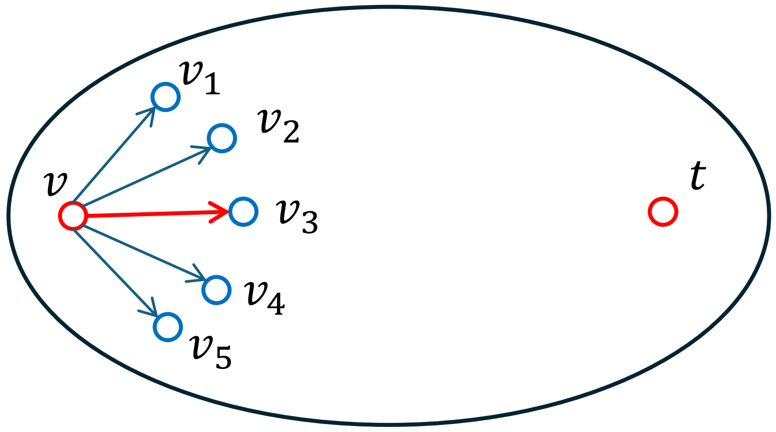

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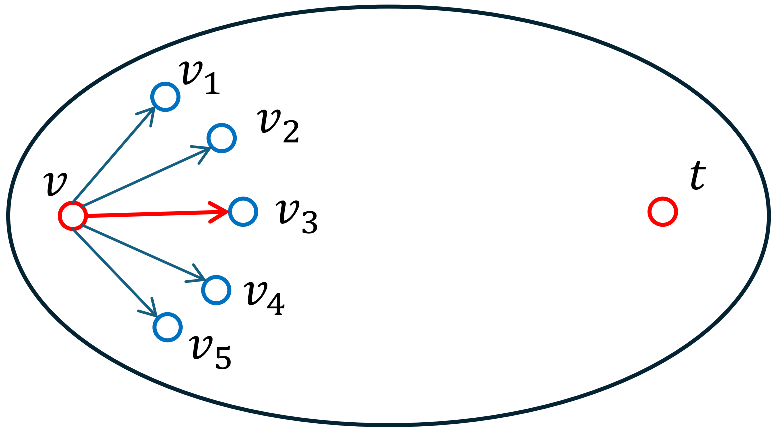

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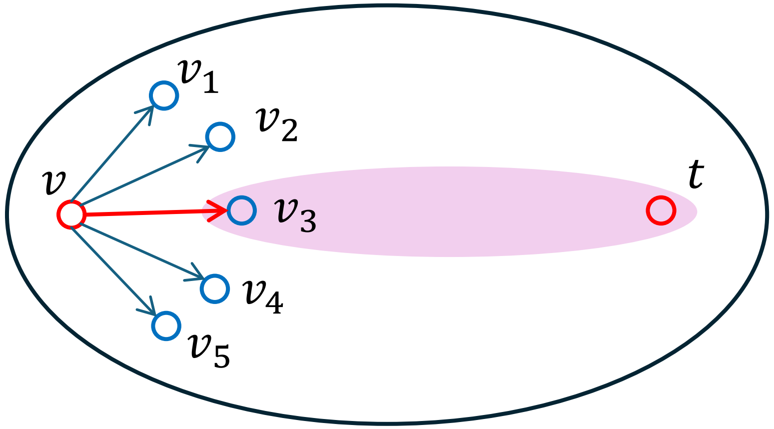

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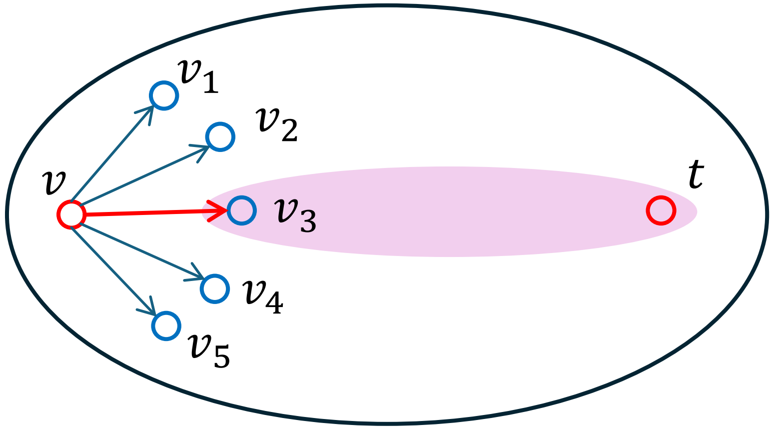

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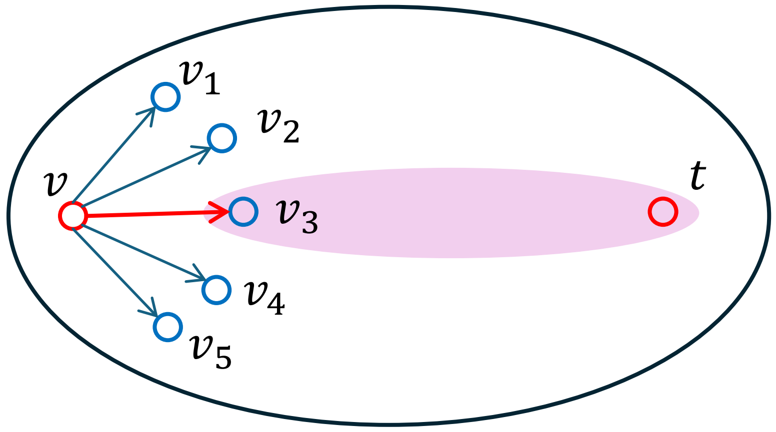

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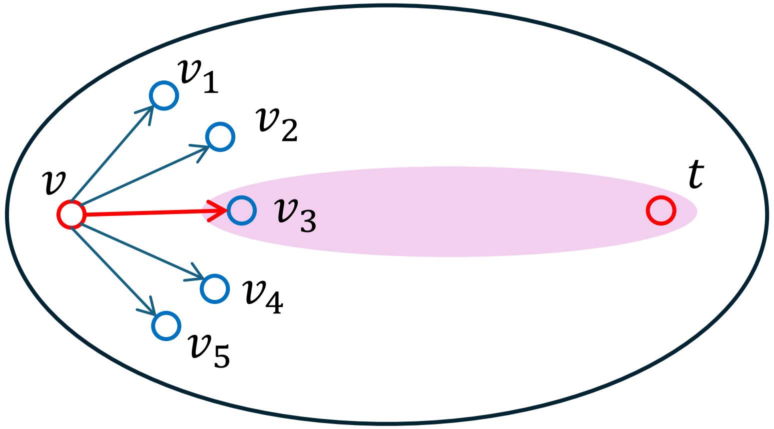

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$$\mathbb{E}[X] = \frac{|\Omega_v|}{\sum_{i=1}^{d} |\Omega_i|} \geq \frac{1}{d} \qquad |\Omega_v| = \left(\sum_{i=1}^{d} |\Omega_i|\right) \mathbb{E}[X]$$

generate ind. X and take average

Estimate the count via Karp-Luby

- A vertex $v \in V$ and a set of ***neighbors*** $v_1, v_2, v_3, \dots, v_d$

$$\Omega_v = \{\text{subgraphs of } G_v \text{ s.t. } v \to t\}$$

- For any neighbor v_i,

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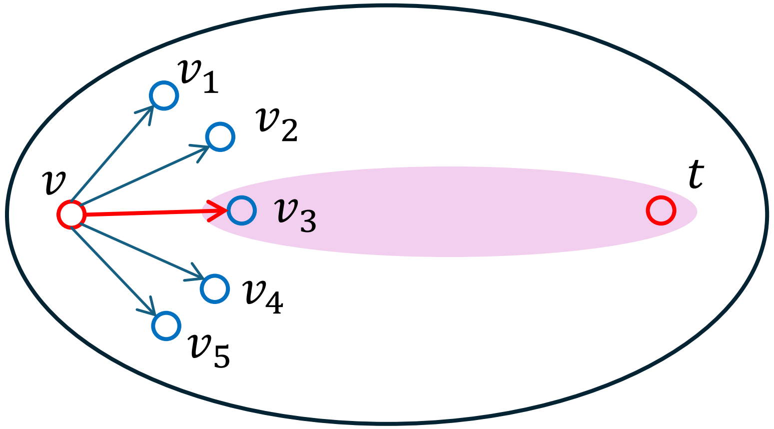
- Vertex v must reach t through some neighbors

$$\Omega_v = \cup_{i=1}^{d} \Omega_i$$



- Apply ***Karp-Luby*** to compute an estimate Z_v such that

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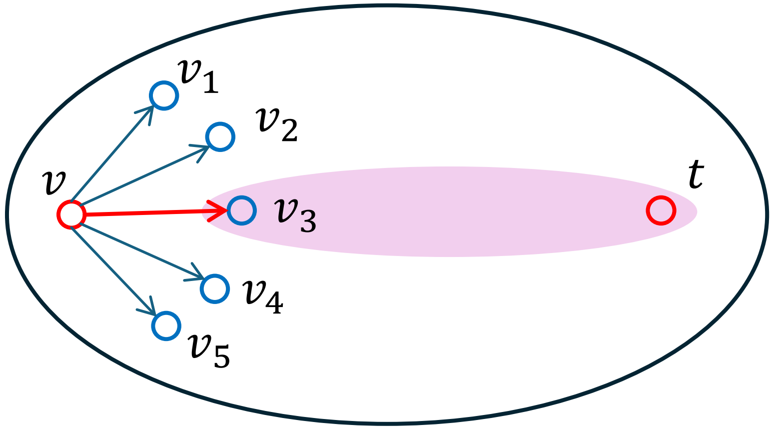

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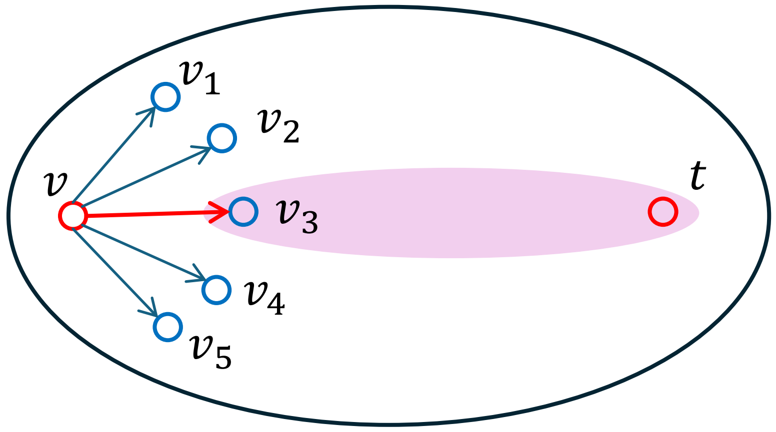

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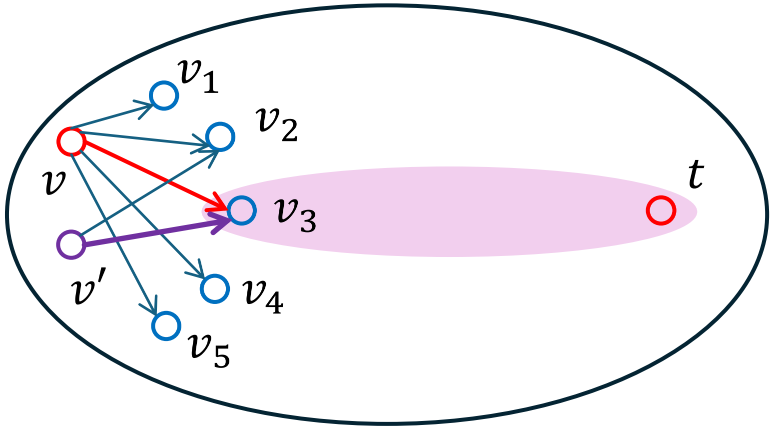

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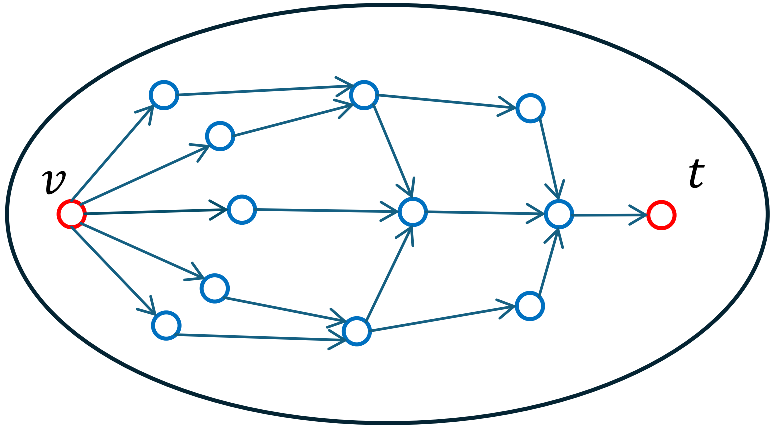
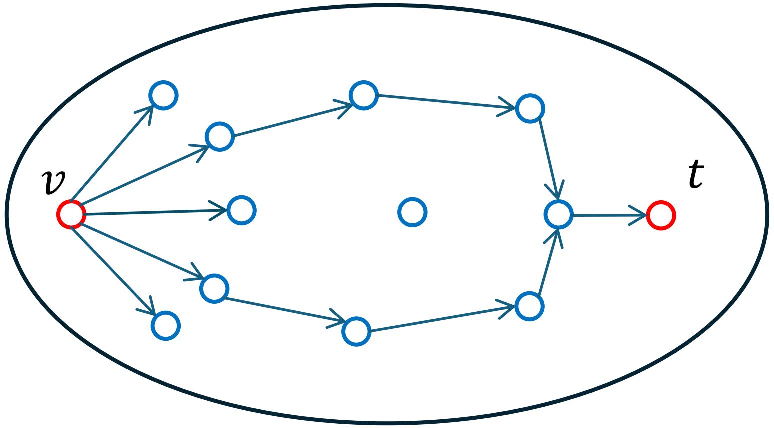
- Different vertices v, v' may have the same neighbor v_i, we ***reuse*** S_{v_i} when computing Z_v and $Z_{v'}$

Generate the samples via self-reduction



graph G_v



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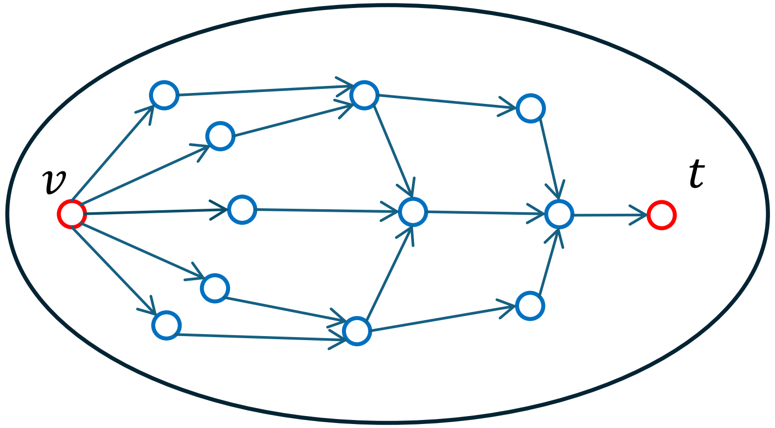

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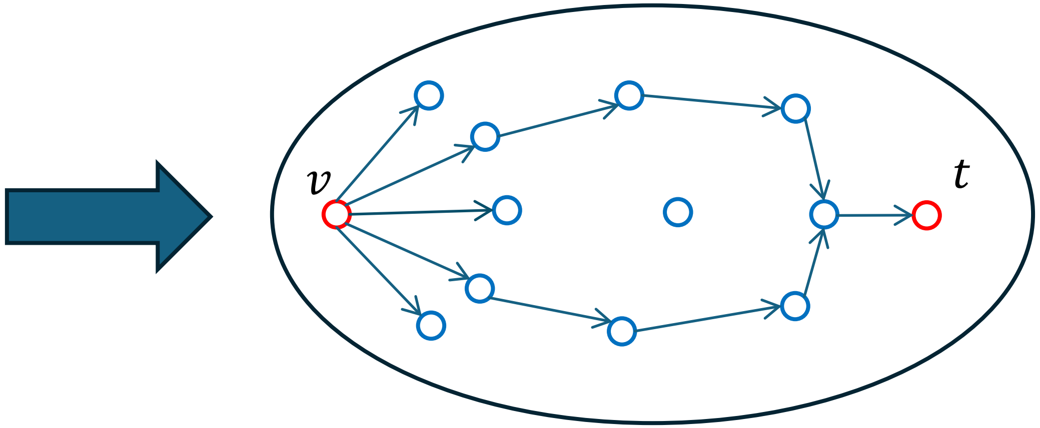

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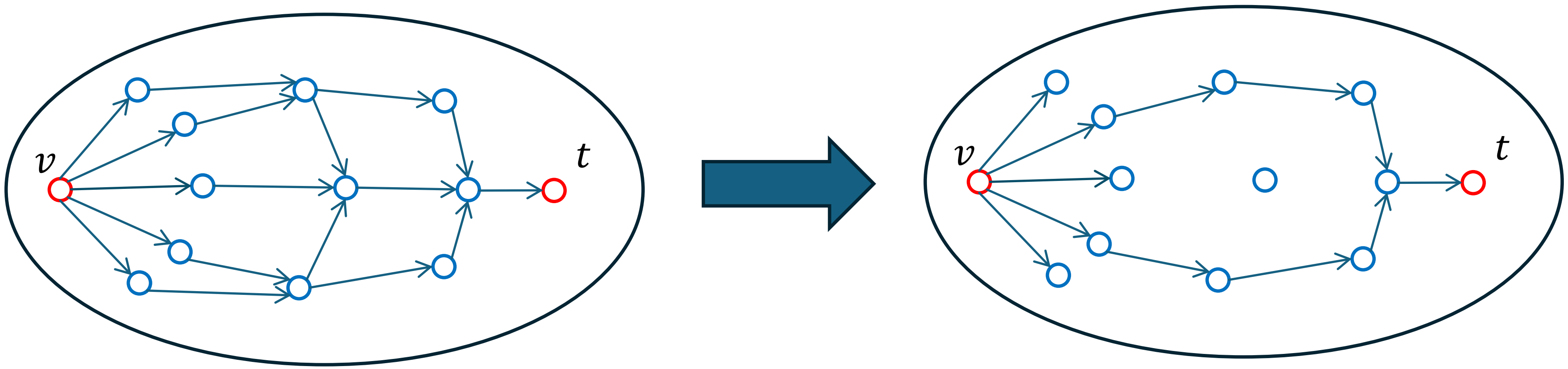

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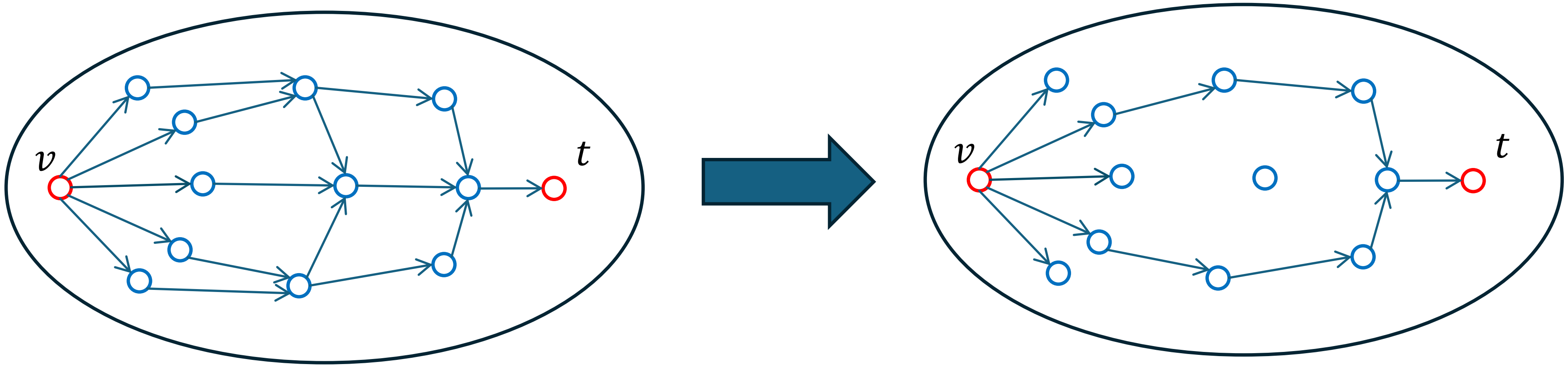

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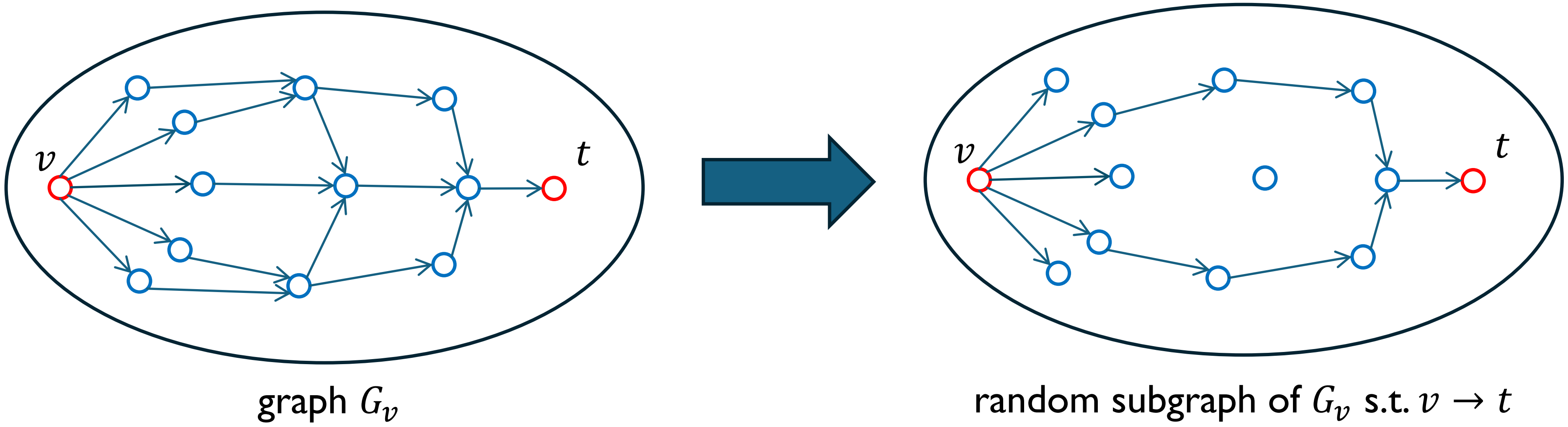

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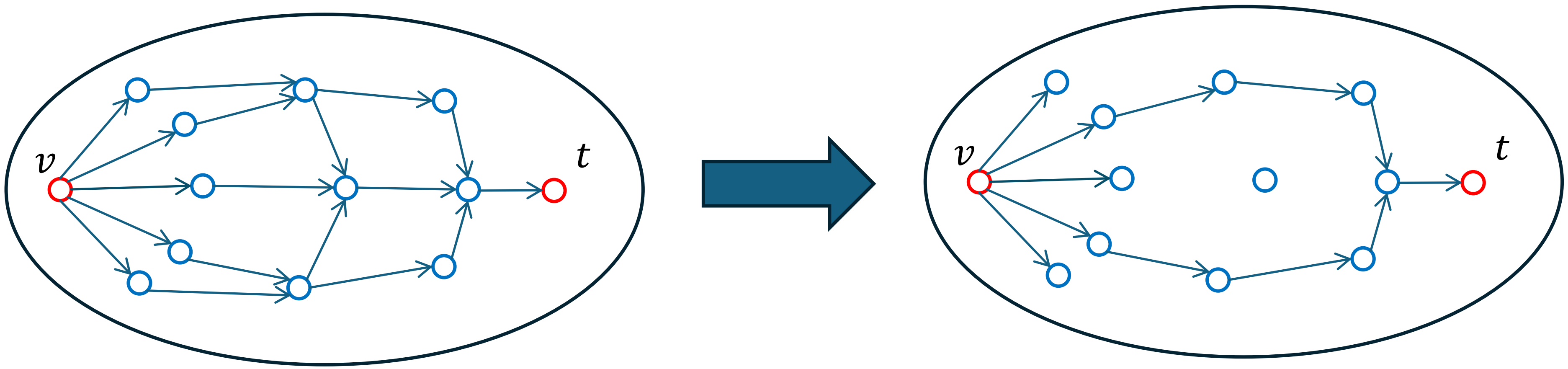

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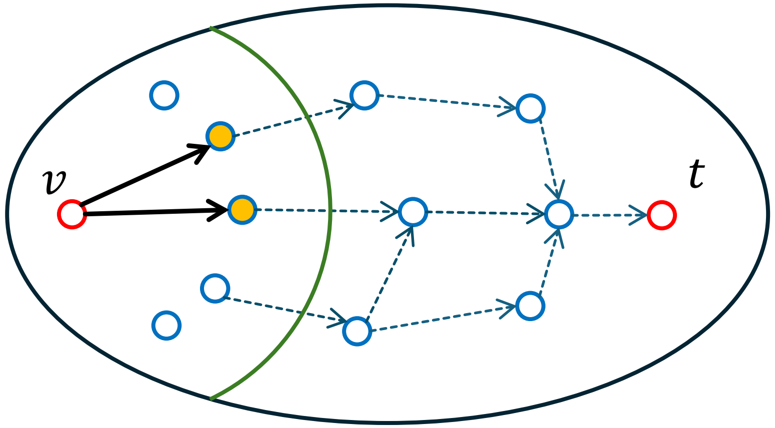
Λ the set of vertices v can reach



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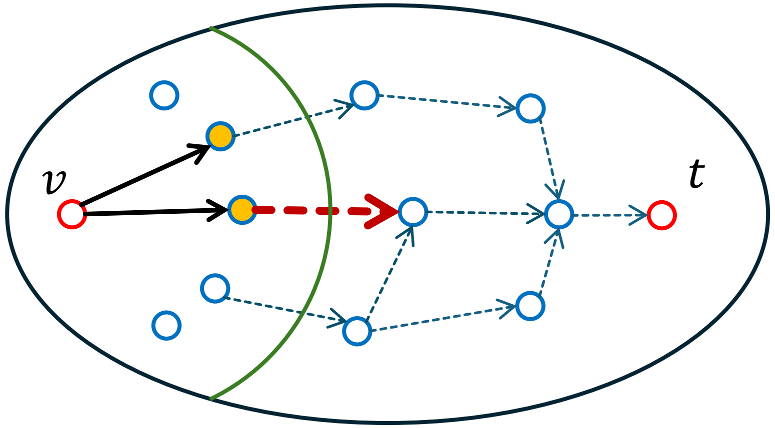

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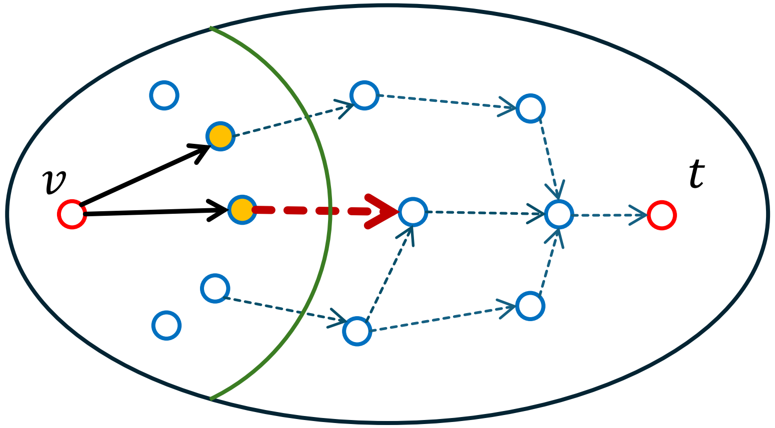

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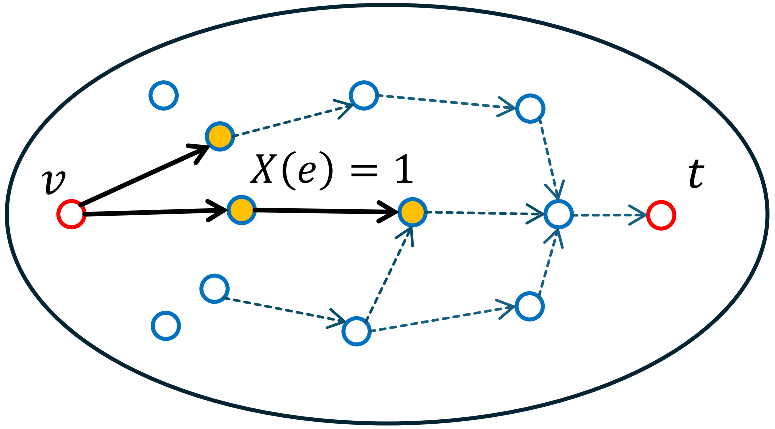
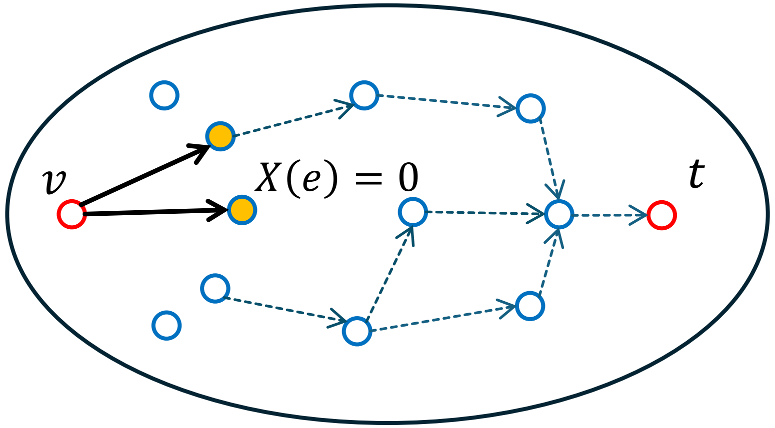


Prob ∝ # {subgraphs v can reach t}

OR



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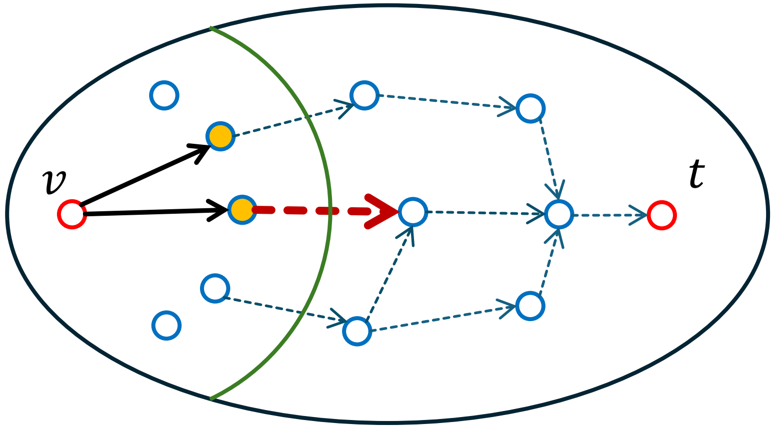

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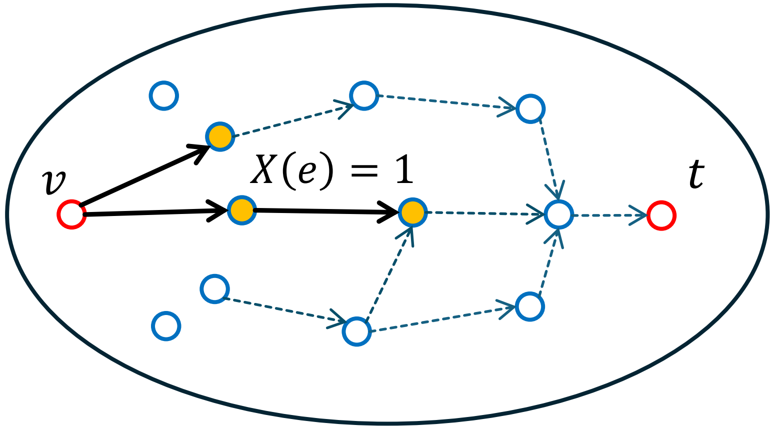

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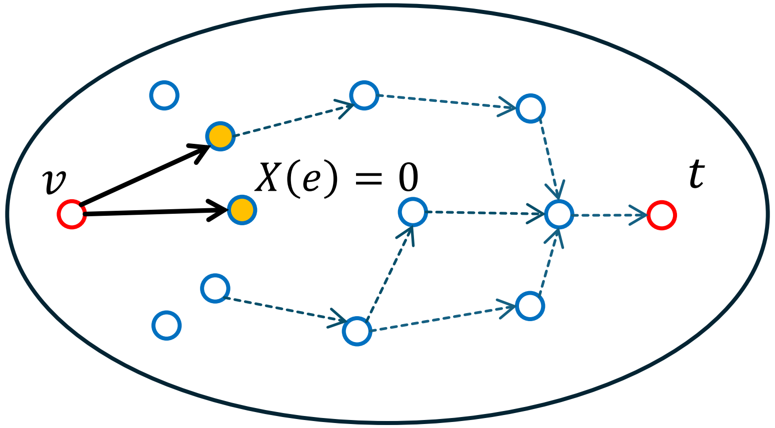

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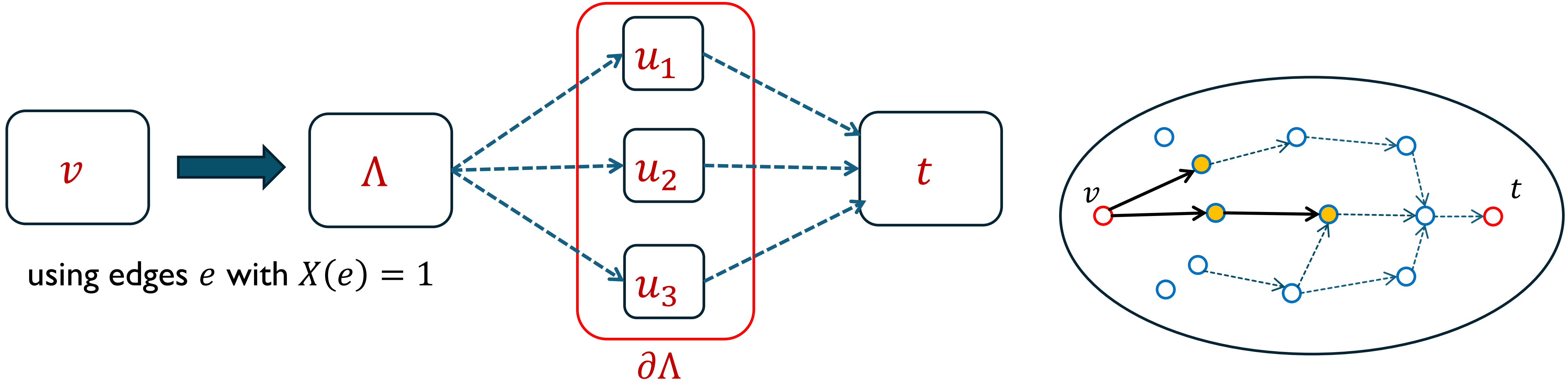
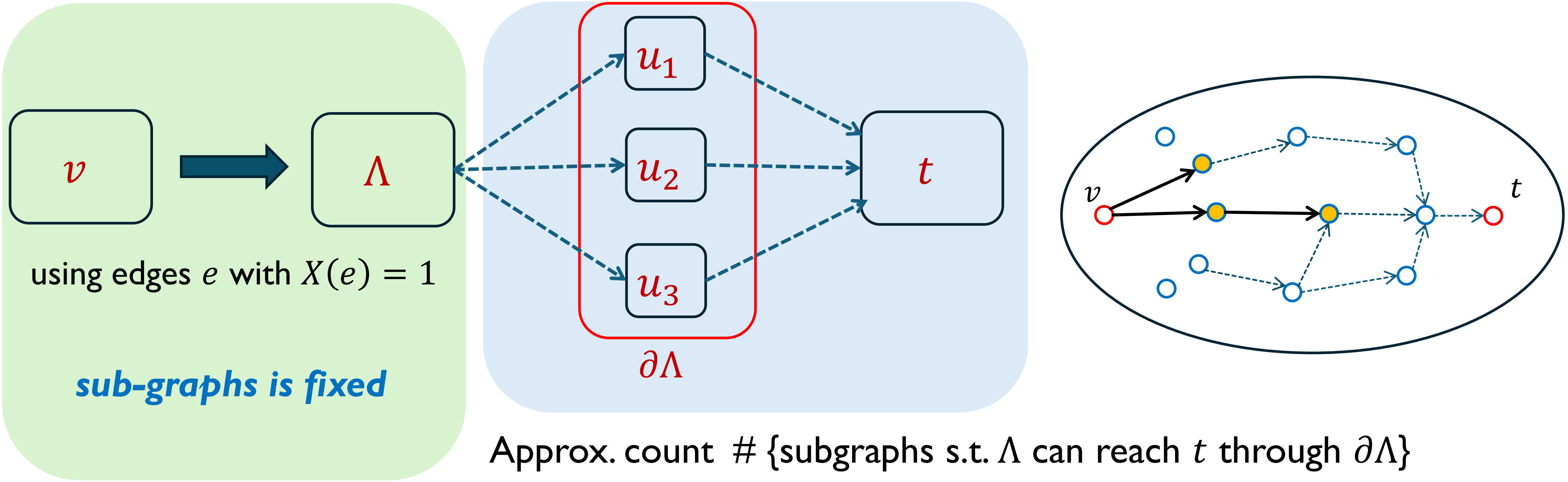
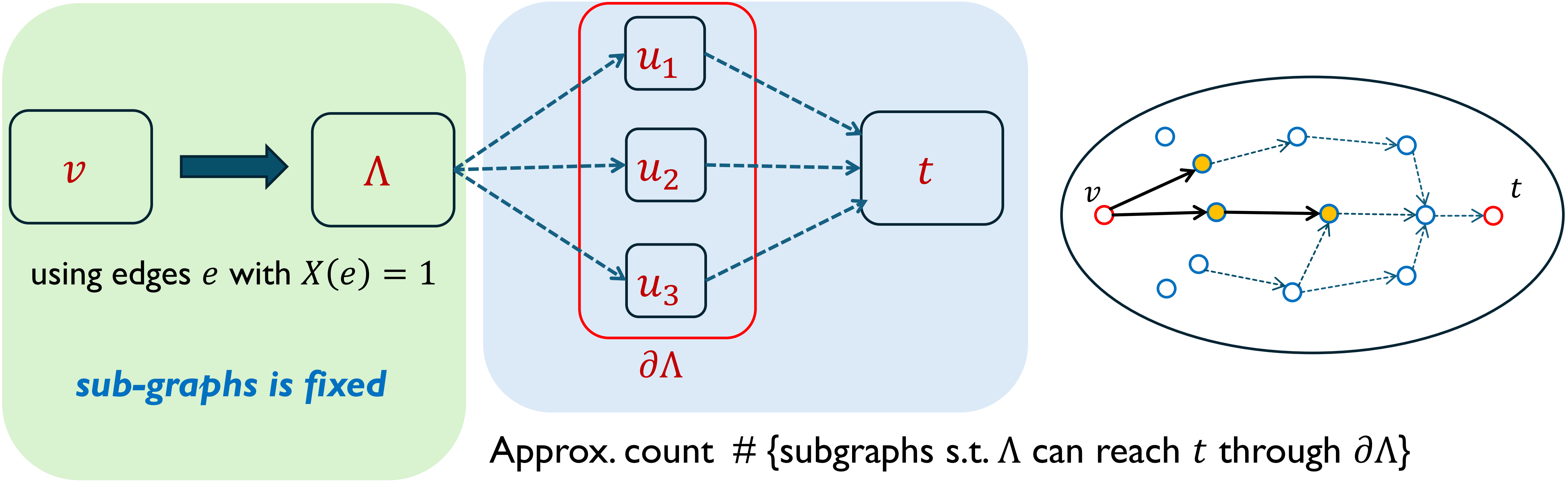


v
Λ
u_1
u_2
u_3
t
$\partial\Lambda$
using edges e with $X(e) = 1$
v
t



v
Λ
using edges e with $X(e) = 1$
sub-graphs is fixed
u_1
u_2
u_3
$\partial\Lambda$
t
Approx. count $\#\{$subgraphs s.t. Λ can reach t through $\partial\Lambda\}$
v
t



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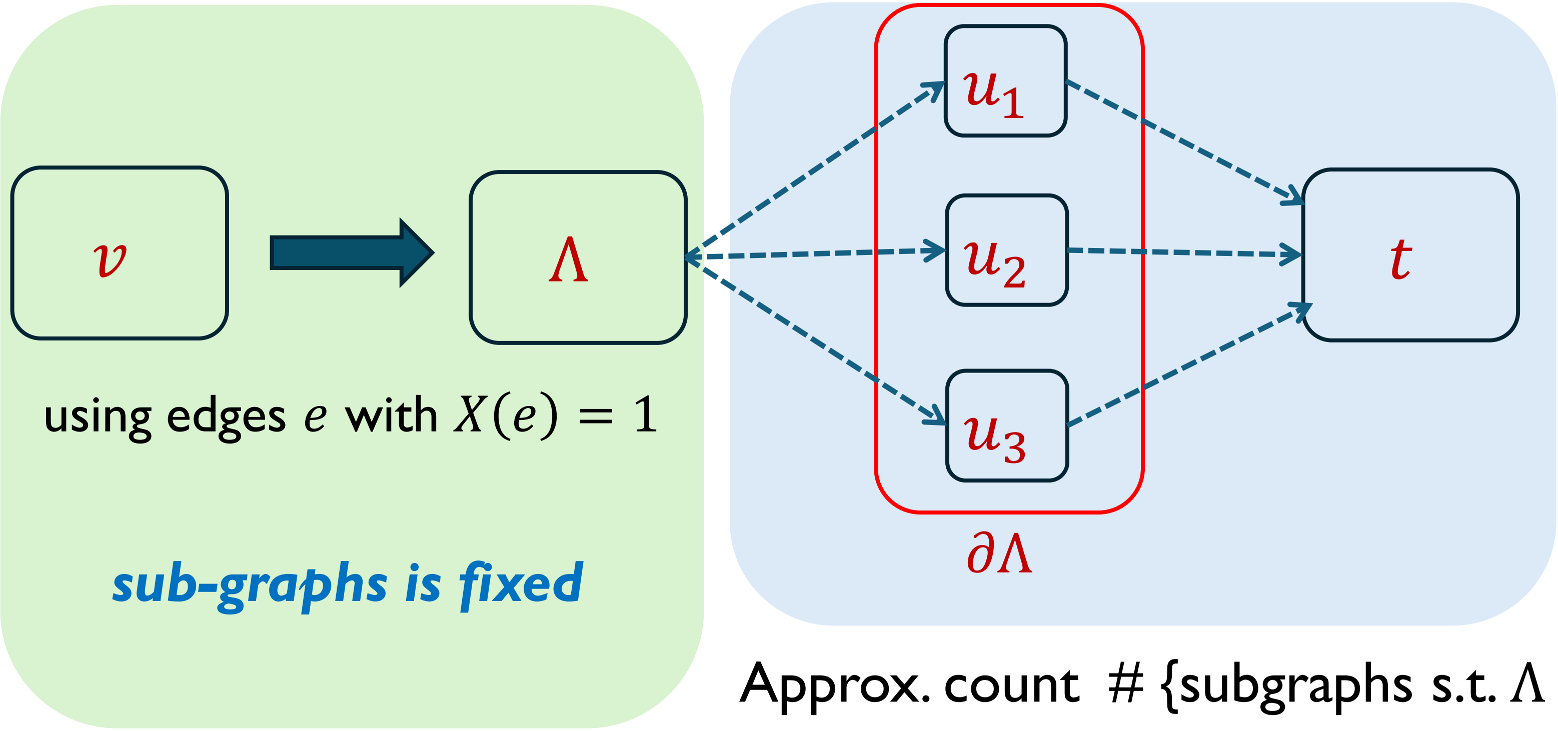
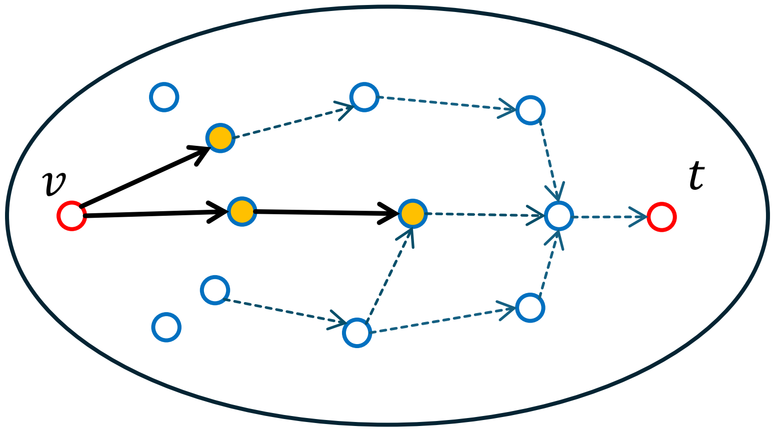


Approx. count $\#$ {subgraphs s.t. Λ can reach t through $\partial\Lambda$}



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- Run ***Karp-Luby algorithm*** by ***reusing*** the samples in nodes at $\partial\Lambda$

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approximate counting
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(reusing samples)

→

estimate ***conditional marginal probability***

→

Generate samples by
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also used for compute the ***count estimate*** Z_v at each node

- Sort all vertices in topological ordering $t = v_1 \prec v_2 \prec \cdots \prec v_n = s$
- G_v: subgraph containing all vertices that can be reached from $v \in V$

$$\Omega_v = \{\text{subgraphs of } G_v \text{ such that } v \text{ can reach } t\}$$

- Z_v: a **count estimate** of the size $\#\Omega_v$
- S_v: a set of **samples**, where each $H \in S_v$ is ***uniform random sample*** from Ω_v

High-level analysis of the algorithm

Challenge: ***reusing*** samples introduces complicated ***correlations***

Key Property: the algorithm only use samples for approx. counting (estimate marginal prob.)

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- Conditional on estimate is correctly only bias random samples with ***exp-small*** error
- Show the whole algorithm is correct by an **induction proof**

Open Problems

- ***Faster*** algorithm for s-t reliability in DAGs

- ***Simple*** algorithm for s-t reliability in DAGs

- FPTAS (***deterministic***) algorithm for s-t reliability in DAGs

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Thank you
Q&A