On the mixing time of Glauber dynamics for the hard-core and related models on $G(n, d/n)$

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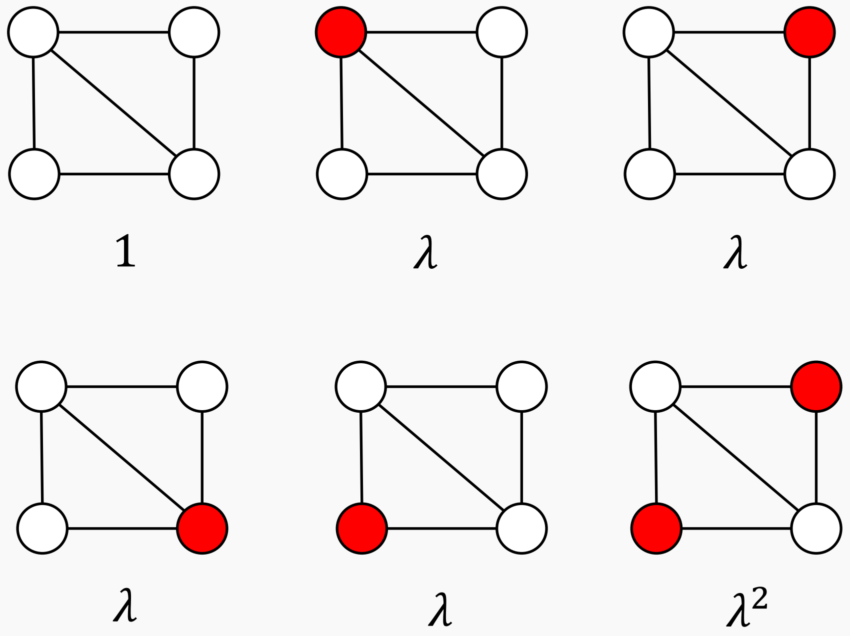
Hardcore model

Graph $G = (V, E)$
- n vertices
- max degree Δ

Fugacity: $\lambda > 0$



Gibbs distribution: $\forall$ ind. set in $S \subseteq V$,

$$\mu(S) = \frac{\lambda^{|S|}}{Z}$$

Partition function

$$Z = \sum_{\text{ind.set } S} \lambda^{|S|}$$

➢ We view the hardcore model as a <u>distribution over $\{0,1\}^V$</u>

Sampling problem and computation phase transition

Input: hardcore model on $G = (V, E)$ with fugacity $\lambda > 0$, error bound $\epsilon > 0$

Output: a random configuration $X \in \{0,1\}^V$

$$\|X - \mu\|_{TV} \le \epsilon$$

$\lambda < \lambda_c$

Poly-time samplers

- Recursion [Weitz 2006]
- Zero-freeness of polynomials [Barvinok 2016] [Patel and Regts, 2017]
- MCMC [Anari, Liu and Oveis Gharan, 2020]

Simplest!

$\lambda > \lambda_c$

No poly-time sampler unless

NP=RP

- [Sly 2010]
- [Sly and Sun, 2012]
- [Galanis, Štefankovič and Vigoda, 2012]

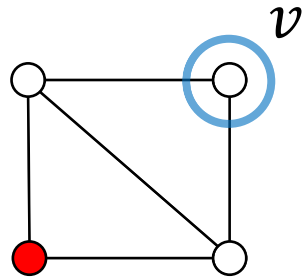
$$\lambda_c(\Delta - 1) = \frac{(\Delta - 1)^{(\Delta - 1)}}{(\Delta - 2)^{\Delta}} \approx \frac{e}{\Delta}$$

MCMC: Glauber dynamics

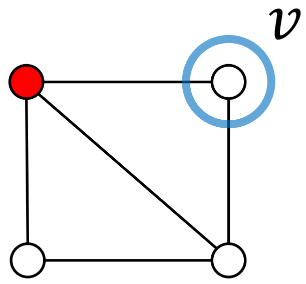
Start from arbitrary ind. set $X \in \{0,1\}^V$

For each t from 1 to T

- Pick $v \in V$ uniformly at random
- Resample $X_v \sim \mu_v(\cdot \mid X_{V-v})$



$\Pr[X_v = 1] = \lambda/(1+\lambda)$
$\Pr[X_v = 0] = 1/(1+\lambda)$



$X_v = 0$

- **Convergence**: $X \sim \mu$ as $T \to \infty$
- **Mixing time**: which $T_{\text{mix}} = T\left(\frac{1}{4e}\right)$ guarantees $\|X - \mu\|_{TV} \le \frac{1}{4e}$
- **Rapid mixing** if $T_{\text{mix}} = \text{poly}(n)$
- **Decay of TV distance**: $T(\epsilon) \le T_{\text{mix}} \log \frac{1}{\epsilon}$

Mixing time of Glauber dynamics

Dobrushin's Condition $\lambda < (1-\delta)\frac{1}{\Delta-1}$ ⇒ $O_\delta(n\log n)$ [Bubley and Dyer, 1997]

Condition $\lambda < (1-\delta)\frac{2}{\Delta-2}$ ⇒ $O_\delta(n\log n)$ [Luby and Vigoda, 1999] [Dyer and Greenhill, 1999]

Uniqueness Condition
$\lambda < (1-\delta)\lambda_c(\Delta-1) \approx (1-\delta)\frac{e}{\Delta}$ ⇒ $O_\delta(n\log n)$ [Anari, Liu and Oveis Gharan, 2020] [Chen, Liu and Vigoda, 2021] [Chen and Eldan, 2022] [Chen, Feng, Yin and Zhang, 2022]

- Mixing of Glauber dynamics: If $\lambda < (1-\delta)\lambda_c(\Delta-1)$ *Work for the Worst Case*
  rapid mixing for ***arbitrary*** hardcore models with max degree Δ
- Hardness for sampling: If $\lambda > \lambda_c(\Delta-1)$,
  there is ***a sequence of graphs with*** max degree Δ s.t. sampling is hard

Hardcore model on random graphs

Question: Fix real numbers $d > 1$ and $\lambda < \lambda_c(d) \approx \frac{e}{d}$. Draw random sample from hardcore model on *Erdős–Rényi* random graph $G\left(n, \frac{d}{n}\right)$

- average degree is d but w.h.p. max degree is $\Delta = \Theta(\frac{\log n}{\log \log n})$, $\lambda \gg \frac{e}{\Delta} \approx \lambda_c(\Delta - 1)$

Previous sampling algorithms: (running time *w.h.p.* over $G(n, d/n)$)

Sampling in time $n^{O(\log d)}$ if $\lambda < \lambda_c(d)$ [Sinclair, Srivastava and Yin, 2013]

- extend Weitz's algorithm to random graphs

Sampling in time $n^{1+\theta}$ if $\lambda < \lambda_c(d)$ [Bezáková, Galanis, Goldberg and Štefankovič, 2022]

- $\theta > 0$ is an arbitrary small constant
- sample from marginals of low-degree vertices then sample configuration for others

Block dynamics $O(n \log n)$ mixing time if $\lambda < \frac{1}{d}$ [Efthymiou, Hayes, Štefankovič and Vigoda 2018]

- Partition vertices into blocks and update block per iteration

Our results

Theorem (Hardcore Model) [Efthymiou and F., This work]

Let $d > 1$ and $\lambda < \lambda_c(d)$ be two real constants. ***W.h.p.*** over $G \sim G\left(n, \frac{d}{n}\right)$.

The ***mixing time*** of ***Glauber dynamics*** is $n^{1+\frac{C}{\log\log n}} = n^{1+o(1)}$, $C = C(\lambda, d)$.

Theorem (Monomer-Dimer Model) [Efthymiou and F., This work]

Let $d > 1$ and $\lambda > 0$ be two constants. ***W.h.p.*** over $G \sim G\left(n, \frac{d}{n}\right)$.

The ***mixing time*** of ***Glauber dynamics*** is $n^{1+o(1)}$.

Monomer-Dimer Model $\forall$ matching $M \subseteq E$, $\mu(M) \propto \lambda^{|M|}$

Proof overview

θ-Fractional Block dynamics $(\mathbf{0} < \boldsymbol{\theta} < \mathbf{1})$

- Pick $S \subseteq V$ with $|S| = \theta n$ uniformly at random
- Resample $X_S \sim \mu_S(\cdot | X_{V-S})$

- Glauber dynamics picks 1 vertex
- Block dynamics picks constant fraction of vertices

Chen-Liu-Vigoda's framework for proving mixing

Spectral Independence (SI)
Marginal lower bound

Mixing of
block dynamics

Mixing of
Glauber dynamics

Marginal lower bound requires $\Delta = O(1)$

Comparison of Markov chains: require $\Delta = O(1)$

One challenge for random graphs: w.h.p. $\Delta = \Theta\left(\frac{\log n}{\log \log n}\right)$

Proof overview

θ-Fractional Block dynamics

- Pick $S \subseteq V$ with $|S| = \theta n$ uniformly at random
- Resample $X_S \sim \mu_S(\cdot |\, X_{V-S})$

- Glauber dynamics picks 1 vertex
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Chen-Liu-Vigoda's framework for proving mixing

Complete SI
A ***mild*** marginal condition

Mixing of
block dynamics

Mixing of
Glauber dynamics

The mild marginal condition holds for hardcore in random graphs
[Chen, Feng, Yin and Zhang, 2022]
[Chen and Eldan, 2022]

Proof overview

θ-Fractional Block dynamics

- Pick $S \subseteq V$ with $|S| = \theta n$ uniformly at random
- Resample $X_S \sim \mu_S(\cdot | X_{V-S})$

- Glauber dynamics picks 1 vertex
- Block dynamics picks constant fraction of vertices

Our Technique for proving mixing

Complete SI
A mild marginal condition → Mixing of ***block*** dynamics → Mixing of ***Glauber*** dynamics

Our contribution

➢ Prove the complete spectral independence for hardcore model in random graphs

Improve the SI result in [Bezáková, Galanis, Goldberg and Štefankovič, 2022]

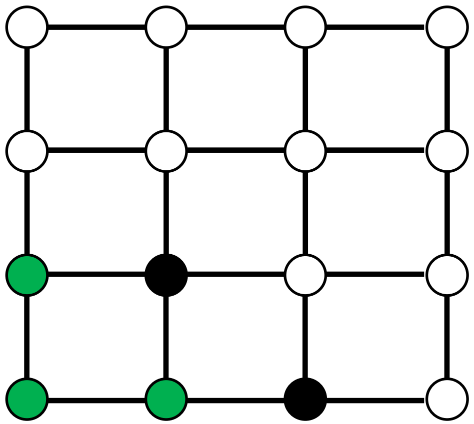
➢ An improved comparison between block and Glauber dynamics

Utilise the SI in the comparison step

Spectral independence

- Gibbs distribution μ over $\{0,1\}^V$
- Influence matrix of μ $\quad \Psi(u,v) = \Pr_{X\sim\mu}[X_v = 1 \mid X_u = 1] - \Pr_{X\sim\mu}[X_v = 1 \mid X_u = 0]$
- η-***spectral independence*** [Anari, Liu and Oveis Gharan, 2020]:

$\lambda_{max}(\Psi) \le \eta$ for μ and ***all conditional distributions*** induced from μ

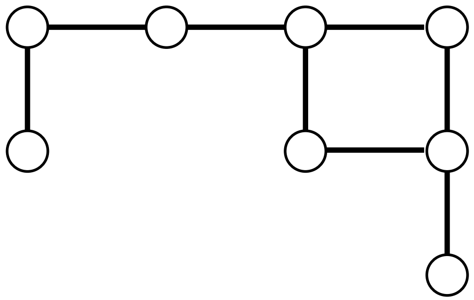


fix a configuration $\sigma \in \{0,1\}^\Lambda$ on Λ

$= 0$ $\quad$ $= 1$

conditional distribution on unfixed vertices $\overline{\Lambda}$

$\mu_{\overline{\Lambda}}(\cdot \mid \sigma)$



hardcore model on a subgraph

Spectral independence

- Gibbs distribution μ over $\{0,1\}^V$
- Influence matrix of μ $\quad \Psi(u,v) = \Pr_{X\sim\mu}[X_u = 1 \mid X_v = 1] - \Pr_{X\sim\mu}[X_u = 1 \mid X_v = 0]$
- η-***spectral independence*** [Anari, Liu and Oveis Gharan, 2020]:

  $\lambda_{max}(\Psi) = \eta$ for μ and ***all conditional distributions*** induced from μ

- η-***complete spectral independence*** [Chen, Feng, Yin and Zhang, 2021]

Complete Spectral Independence	=	Spectral Independence under certain **external field**

We focus on the **Spectral Independence**, complete SI follows from a similar proof

Spectral independence in random graphs

Previous SI result for marginal [Bezáková, Galanis, Goldberg and Štefankovič, 2022]

If $d > 1$ and $\lambda < \lambda_c(d)$. ***W.h.p.*** over $G \sim G\left(n, \frac{d}{n}\right)$,

μ_S is $O(\log n)$-spectrally independent, μ_S is the **marginal** of low degree vertices

➡ A sampling ***algorithm*** with $n^{1+\theta}$ running time, $\theta > 0$ is an arbitrary small constant

Our SI result for Gibbs distribution [Efthymiou and F., This work]

If $d > 1$ and $\lambda < \lambda_c(d)$. ***W.h.p.*** over $G \sim G\left(n, \frac{d}{n}\right)$,

μ is $(\log n)^c$-spectrally independent for some constant $c < 1$

➡ $n^{1+C/\log\log n} = n^{1+o(1)}$ mixing time for ***Glauber dynamics***

Spectral Independence via total influence

$$\lambda_{\max}(\Psi) \le \|\Psi\|_\infty = \max_{v\in V} \sum_{u\in V} |\Psi(v,u)| \le \eta$$

Our method [Efthymiou and F., 2023]

- Find a ***diagonal matrix*** D such that

$$D(v,v) = \deg(v)^c, \qquad \text{where } c \in \left(\frac{1}{2}, 1\right)$$

- Establish SI via the ***weighted total influence*** $\lambda_{\max}(\Psi) \le \|D^{-1}\Psi D\|_\infty \le \eta$

$$\forall v, \qquad \sum_{u\in V} |\Psi(v,u)| \deg(u)^c \le \eta \deg(v)^c$$

If the $\deg(v)$ is large,
v may cause large total influence

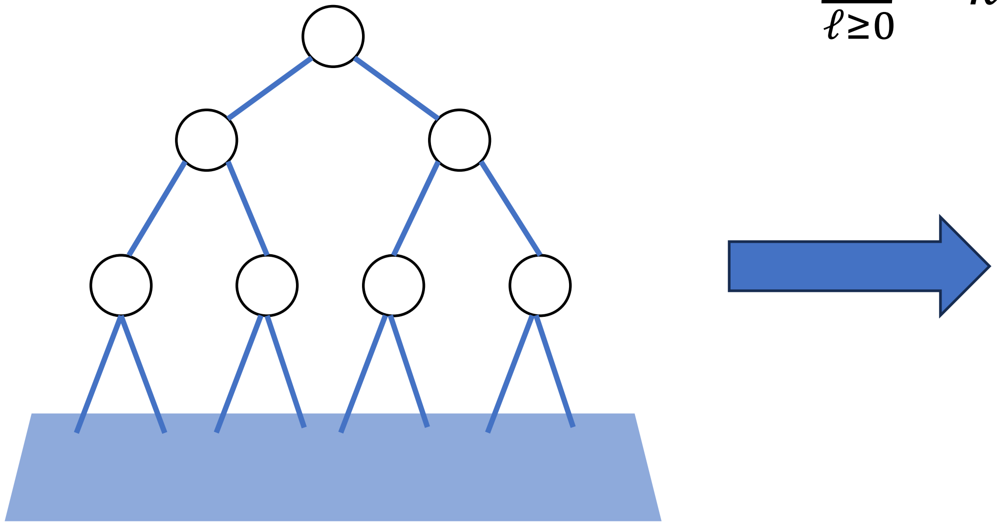
$\deg(u)^c$ would make LHS large
In ***"average"***, $\deg(u)$ is small

$\deg(v)^c$ in RHS is also large

***k*-branching factor** [Bezáková, Galanis, Goldberg and Štefankovič, 2022]

In $G(V, E)$, for any v, $N(v, \ell) = \#\{\text{simple paths from } v \text{ of length } \ell\}$

$$S_k = \sum_{\ell \geq 0} \frac{N(v, \ell)}{k^\ell}$$



d-ary tree

$\forall k > d,$

$$S_k = \sum_{\ell \geq 0} \frac{d^\ell}{k^\ell} = O_{d,k}(1)$$

Example: branching factor in d-ary tree

- In random graph, we use branching factor to measure how much influence that can percolate from v to all other vertices.

k-branching factor [Bezáková, Galanis, Glodberg and Štefankovič, 2022]

In $G(V,E)$, for any v, $N(v,\ell) = \#\{\text{simple paths from } v \text{ of length } \ell\}$

$$S_k = \sum_{\ell \geq 0} \frac{N(v,\ell)}{k^\ell} \quad \text{(A notion of average degree)}$$

Branching factor in random graphs [Bezáková, Galanis, Goldberg and Štefankovič, 2022]

$$\forall k > d, \qquad \text{w.h.p. over } G\left(n, \frac{d}{n}\right), \qquad S_k \leq \log n$$

η-Spectral independence for hardcore in random graphs [Efthymiou and F., 2023]

$$\text{If } \lambda < \lambda_c(d), \qquad \text{w.h.p. over } G\left(n, \frac{d}{n}\right), \qquad \eta \leq S_{d+\epsilon}^c = (\log n)^c, \qquad c = c(\lambda, d) < 1$$

- Reduce the graph to a ***self-avoiding-walk tree*** (SAW-tree) [Weitz 2006]
- Analyse the weighted influence in SAW-tree via ***correction decay*** technique
  [Sinclair, Srivastava, Štefankovič and Yin 2017] [Bezáková, Galanis, Goldberg and Štefankovič, 2022]

Summary and Open Problems

Mixing results

- $n^{1+o(1)}$ mixing time for hardcore model on $G(n, d/n)$ when $\lambda < \lambda_c(d)$
- $n^{1+o(1)}$ mixing time for monomer-dimer model on $G(n, d/n)$

Spectral independence result

- $(\log n)^c$-spectral independence for hardcore model on $G(n, d/n)$ when $\lambda < \lambda_c(d)$

More distributions in random graphs

- Sample q-colourings in $G\left(n, \frac{d}{n}\right)$
  - Current best result is the $O\left(n^{2+1/\log d}\right)$ mixing when $q > 1.763d$ [Efthymiou, Hayes, Štefankovič and Vigoda 2018]

Thank you!