# Field dynamics: a new tool to boost mixing results

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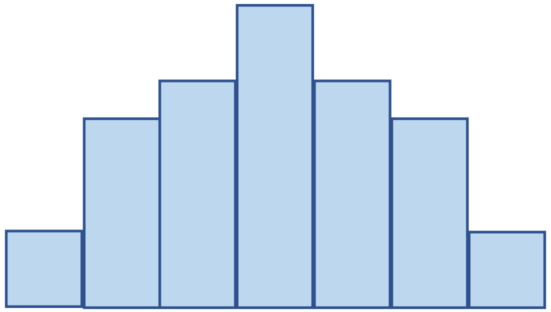
# Sampling, counting and phase transition

Boolean variables set $V$, weight function $w: \{-,+\}^V \to \mathbb{R}_{\geq 0}$

**joint distribution** $\mu$:

$$\forall X = (X_v)_{v \in V} \in \{-,+\}^V, \qquad \mu(X) \propto w(X)$$

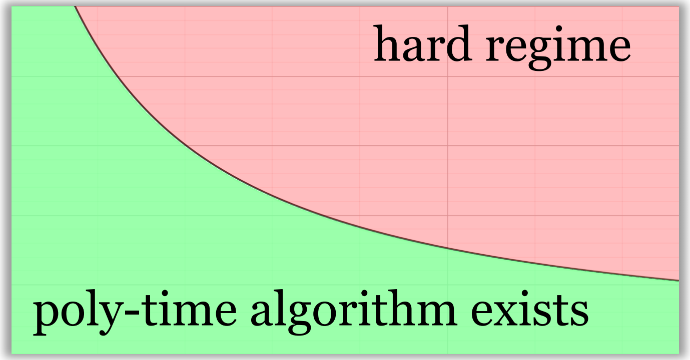


**Sampling problem**

Draw (approximate) random samples from distribution $\mu$

**Goal:**

Prove ***optimal*** mixing results up to the computational phase transition threshold



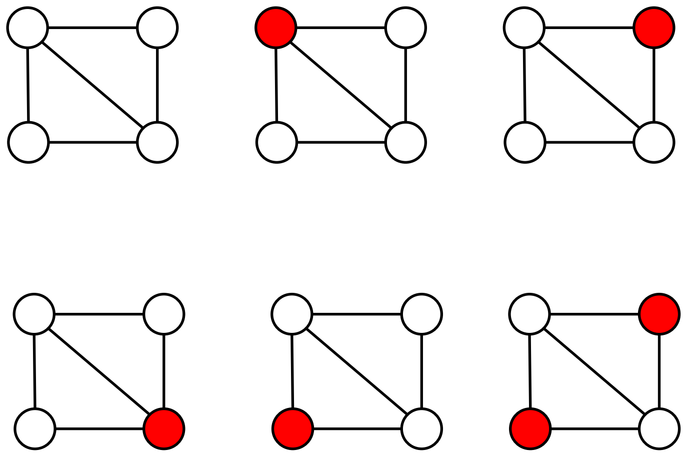
# Example: Hardcore model

- graph $G = (V, E)$, parameters $\lambda$;
- Gibbs distribution $\mu$:

$$\forall \text{independent set } I \subseteq V, \qquad \mu(I) \propto \lambda^{|I|}.$$

- Equivalent state space of $\mu$:

$$\{-,+\}^V = \{\text{occupied}, \text{unoccupied}\}^V$$



**Computational phase transition**

- $\lambda < \lambda_c(\Delta)$: poly-time algorithm for sampling[Weitz06]
- $\lambda > \lambda_c(\Delta)$: no poly-time algorithm unless $\boldsymbol{NP} = \boldsymbol{RP}$ [Sly10]

$$\lambda_c(\Delta) = \frac{(\Delta - 1)^{(\Delta-1)}}{(\Delta - 2)^{\Delta}} \approx \frac{e}{\Delta}$$

# Glauber dynamics for hardcore model

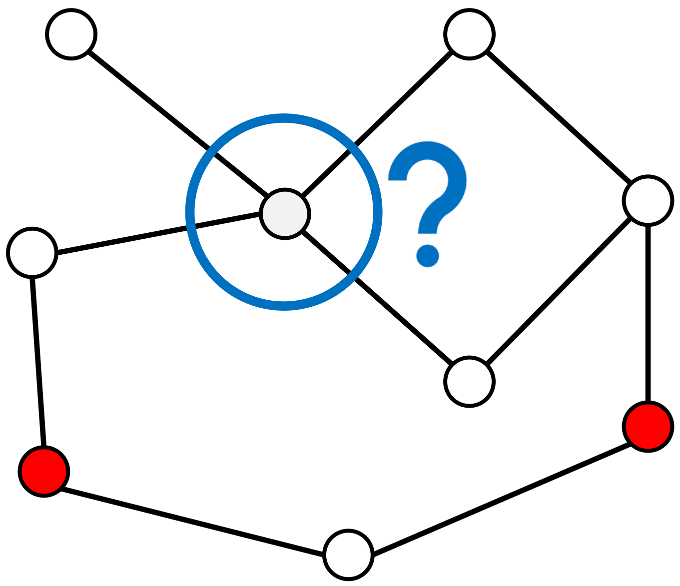
Start from an arbitrary independent set $X$;

**For** each transition step **do**

- Lazy w.p. $\frac{1}{2}$, otherwise do as follows:
- Pick a vertex $v$ uniformly at random;
- **If** $X_u = -$ for all neighbors $u$ **then**

$$X_v = \begin{cases} + & \text{w.p.} \quad \lambda/(1+\lambda) \\ - & \text{w.p.} \quad 1/(1+\lambda) \end{cases}$$

- **Else** $X_v \leftarrow -$




**Mixing time**: $T_{\mathrm{mix}} = \max_{X_0 \in \Omega} \min \left\{ t \mid d_{TV}(X_t, \mu) \le \frac{1}{4e} \right\}$,

$d_{TV}(X_t, \mu)$: the *total variation distance* between $X_t$ and $\mu$.

# Previous works

| Work | Condition | Mixing Time |
|---|---|---|
| Dobrushin 1970 | $\lambda \leq \frac{1-\delta}{\Delta - 1}$ | $O\left(\frac{1}{\delta} n \log n\right)$ |
| Luby, Vigoda, 1999 | $\lambda \leq \frac{2(1-\delta)}{\Delta - 2}$ | $O\left(\frac{1}{\delta} n \log n\right)$ |
| Efthymiou *et al*, 2016 | $\lambda \leq (1-\delta)\lambda_c(\Delta)$<br>$\Delta \geq \Delta_0(\delta), \text{girth} \geq 7$ | $O\left(\frac{1}{\delta} n \log n\right)$ |

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| Anari, Liu, Oveis Gharan, 2020<br>improved by Chen, Liu, Vigoda, 2020 | $\lambda \leq (1-\delta)\lambda_c(\Delta)$ | $n^{O(1/\delta)}$ |
| Chen, Liu, Vigoda, 2021 | $\lambda \leq (1-\delta)\lambda_c(\Delta)$ | $\Delta^{O(\Delta^2/\delta)} n \log n$ |

**Open question**:

Can we prove the fast (optimal) mixing for all degrees ?

# Mixing time of Glauber dynamics when $\lambda \le (1-\delta)\lambda_c$

| Work | Mixing Time | Technique |
|---|---|---|
| Anari, Liu, Oveis Gharan, 2020<br>improved by Chen, Liu, Vigoda, 2020 | $n^{O(1/\delta)}$ | Spectral Independence (SI) |
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| (2) **Chen, F., Yin, Zhang, 2022** | $e^{O(1/\delta)} n \log n$ | Entropic Independence (EI) & Field Dynamics |
| (3) Chen, Eldan, 2022 | $e^{O(1/\delta)} n \log n$ | Localization Scheme |

(1) Zongchen Chen (2) Xiaoyu Chen (3) Yuansi Chen

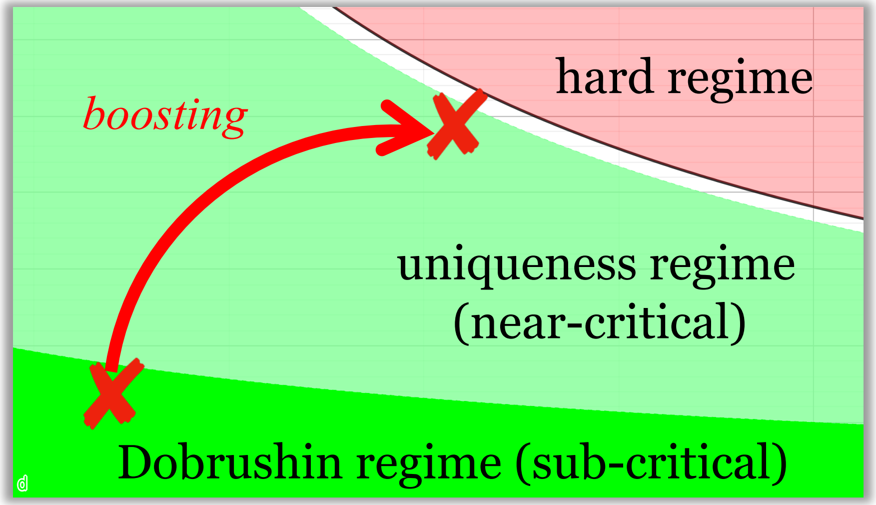
## Hardcore model in uniqueness regime

If $\lambda$ is ***close*** to $\lambda_c(\Delta)$, e.g., $\lambda = 0.999\lambda_c$ (**near-critical**)

analyzing mixing time is ***hard***

- If $\lambda$ is ***far-away*** from $\lambda_c(\Delta)$, e.g., $\lambda \leq 0.1\lambda_c$ (**sub-critical**)

analyzing mixing time is ***easy***




## Boosting Theorem

**Boosting** mixing results from **sub-critical regime** to **near-critical regime**

- Boost ***spectral gap*** of Glauber dynamics [CFYZ21]
- Boost ***modified log-Sobolev constant*** of Glauber dynamics [CFYZ22]

*Proved by a new Markov chain: field dynamics*

# Revisit Chen-Liu-Vigoda's technique

**Simpler task**: poly-time sampling algorithm

**Input**: hardcore model with $\lambda \le (1-\delta)\lambda_c(\Delta)$ and $\Delta$ can be unbounded;

**Output**: random sample $X$ s.t. $d_{TV}(X,\mu) = \frac{1}{\text{poly}(n)}$.

## $\theta$-fractional block dynamics

Parameter: $\theta \in (0,1)$

Initialization: arbitrary $X \in \{-,+\}^V$

Update: for each $t = 1$ to $T$

- pick $S \subseteq V$ with $|S| = \theta n$ u.a.r.;
- $X_S \sim \mu_S(\cdot \mid X_{V\setminus S})$;

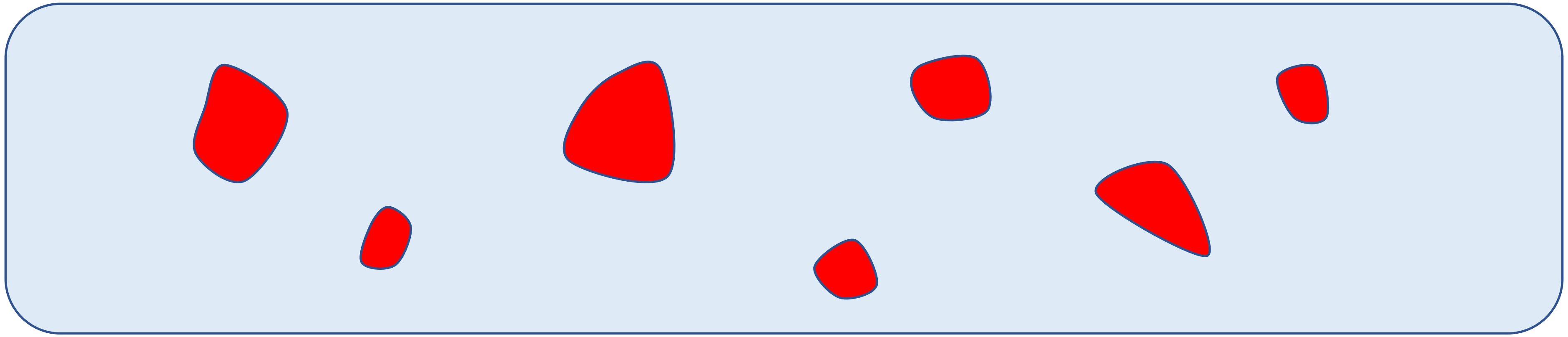
## Mixing result [CLV21]

$$\lambda_{\text{gap}} \ge \theta^{O(1/\delta)}$$

$$d_{TV}(\mu, X) \le \frac{1}{\text{poly}(n)} \text{ if } T = \left(\frac{1}{\theta}\right)^{O(1/\delta)} n \log n$$

**Question**: how to ***efficiently*** simulate the transition step $X_S \sim \mu_S(\cdot \mid X_{V\setminus S})$?

Update step $X_S \sim \mu_S(\cdot \mid X_{V\setminus S})$: sample from hardcore model $(G[S], \lambda)$ with boundary condition $X_{V\setminus S}$



**Observation** [Chen, Liu and Vigoda, 2021]

If $\theta = O\left(\frac{1}{\Delta}\right)$, then w.h.p., $G[S]$ is a set of small connected components

**Mixing time**

$\theta = O(1/\Delta)$-fractional block dynamics

$T = \Delta^{O(1/\delta)}\, n \log n$ steps

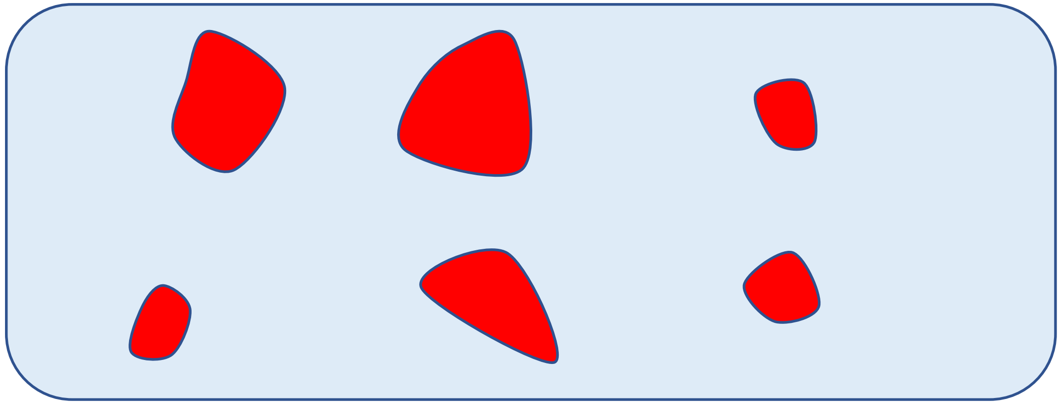
**Simulation of each step**

**brute force** on each component

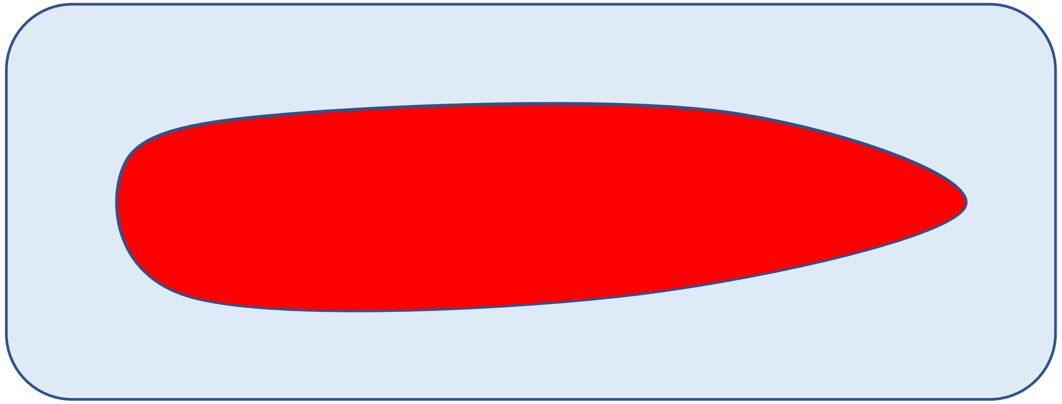
Expected cost $= O(n)$

Total expected running time of the algorithm: $\Delta^{O(1/\delta)} n^2 \log n$

Natural idea: set $\theta = \frac{1}{100}$ → Mixing time $T = 2^{O(1/\delta)} n \log n = O_\delta(n \log n)$



$\theta = O(1/\Delta)$



$\theta = O(1)$

**Issue** : how to sample from hardcore model $(G[S], \lambda)$ with boundary condition $X_{V \setminus S}$?

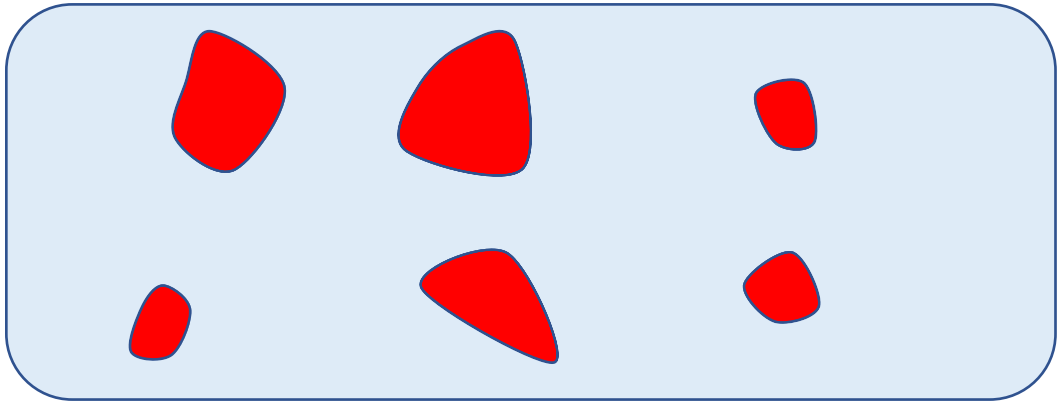
**Observation** [Chen, F. Yin and Zhang, 2021]

The maximum degree of $G[S]$ can be small.

For any $v \in S$,

$$\mathbb{E}[\text{degree of } v \text{ in } G[S]] \approx \theta \deg_G(v) = \frac{\deg_G(v)}{100}$$

Natural idea: set $\theta = \frac{1}{100}$ → Mixing time $T = 2^{O(1/\delta)} n \log n = O_\delta(n \log n)$



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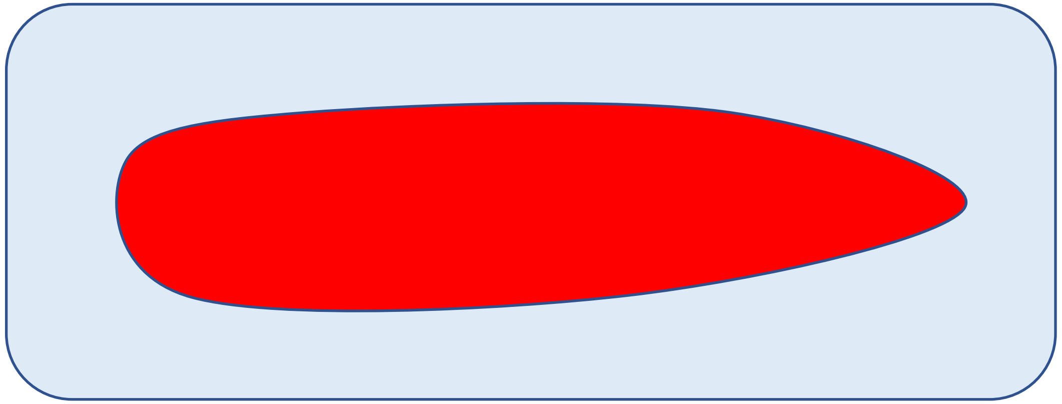


$\theta = O(1)$

**Issue** : how to sample from hardcore model $(G[S], \lambda)$ with boundary condition $X_{V \setminus S}$?

$$\lambda \leq \lambda_c(\Delta_G) \approx \frac{e}{\Delta_G}$$

uniqueness condition in $(\lambda, \Delta_G)$

If we can show

$$\Delta(G[S]) \ll \Delta_G$$

the $(G[S], \lambda)$ is **easy** to sample from

**Case 1**: the maximum degree $\Delta_G$ of original graph $G$ satisfies $\Delta_G \geq 100 \log n$

- By **concentration**, for any $v \in S$, expected degree $\leq \Delta_G/100$,

$$\Pr\left[\text{degree of } v \text{ in } G[S] \leq \frac{\Delta_G}{10}\right] \geq 1 - \frac{1}{n^{10}}$$

- Bound a **union bound,** w.h.p. (prob $\geq 1 - \frac{1}{n^7}$)

  In every transition step, the maximum degree of $G[S]$ is at most $\frac{\Delta_G}{10}$

- The hardcore model $(G[S], \lambda)$ satisfies **Dobrushin's condition**

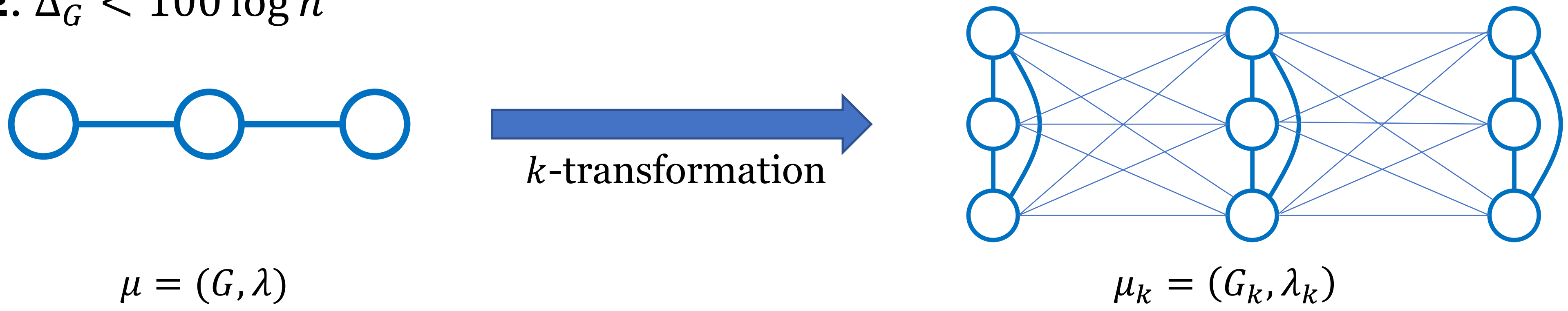
  simulate the Glauber dynamics for $O(n \log n)$ steps.

$\theta = 1/100$-fractional block dynamics
$T = 2^{O(1/\delta)} n \log n$ steps

simulation cost of each step
$O(n \log n)$

Total running time of the algorithm: $2^{O(1/\delta)} n^2 \log^2 n$

**Case 2**: $\Delta_G < 100 \log n$

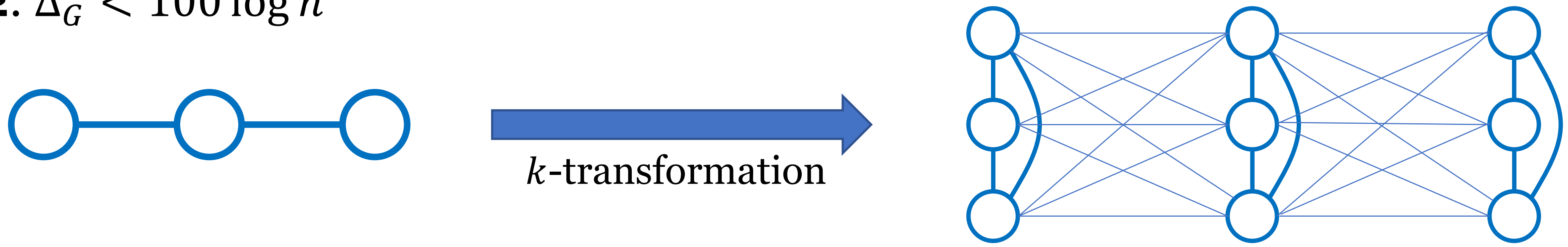



- Graph $G_k$: $v \in V$ ⟶ size $k$ clique $C_v$; $\{u, v\} \in E$ ⟶ connect $C_u$ and $C_v$;
- Parameter: $\lambda_k = \lambda / k$;

## Properties of the $k$-transformation:

- $\mu_k$ is $O(1/\delta)$ **spectrally independence** ⟶ block dynamics on $\mu_k$ is **rapid mixing** [CLV21]
- if $k = \Omega(\log n)$, the **max degree** of $G_k$ is **large** ⟶ $\mu_k = (G_k, \lambda_k)$ is in **Case 1**
- if $X \sim \mu_k$, then $X' = f_k(X) \sim \mu$

**Case 2**: $\Delta_G < 100 \log n$




$\mu = (G, \lambda)$ $\mu_k = (G_k, \lambda_k)$

- Graph $G_k$: $v \in V$ ⟶ size $k$ clique $C_v$; $\{u, v\} \in E$ ⟶ connect $C_u$ and $C_v$;
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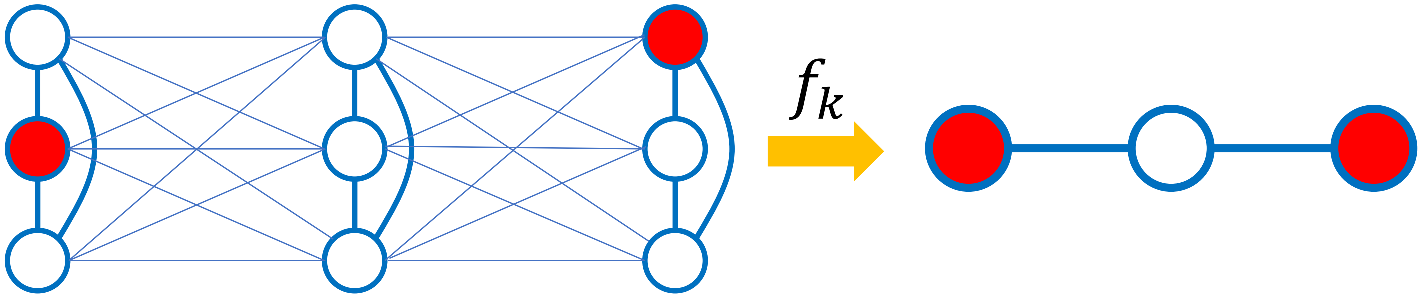
- if $k = \Omega(\log n)$, the **max degree** of $G_k$ is **large** ⟶ $\mu_k = (G_k, \lambda_k)$ is in **Case 1**
- if $X \sim \mu_k$, then $X' = f_k(X) \sim \mu$
  - $X'_v = +$ if $\exists u \in C_v$ s.t. $X_u = +$
  - $X'_v = -$ if $\forall u \in C_v$, $X_u = -$



## Algorithm for sampling from the hardcore model

**Input**: graph $G = (V, E)$ and parameter $\lambda \leq (1-\delta)\lambda_c(\Delta)$

- ➢ apply $k = \Omega(\log n)$-transformation to get $\mu_k$;
- ➢ simulate $\left(\theta = \frac{1}{100}\right)$-fractional **block dynamics** $(X_t)_{t=0}^{T}$ on $\mu_k$:
  - $T = 2^{O(1/\delta)}(nk)^2 \log(nk)$;
  - every transition is simulated by an $O(nk \log(nk))$-step **Glauber dynamics**
- ➢ output $f_k(X_T)$

$\left(f_k(X_t)\right)_{t=0}^{T}$ $\xrightarrow{k \to \infty}$ **Field Dynamics**

Apply the mapping $f_k$ on every step of block dynamics

New Markov chain for sampling from $\mu = \text{Hardcore}(G, \lambda)$

# ***Field Dynamics***

**Input**: hardcore model $\mu = (G, \lambda)$, a parameter $\theta \in (0,1)$

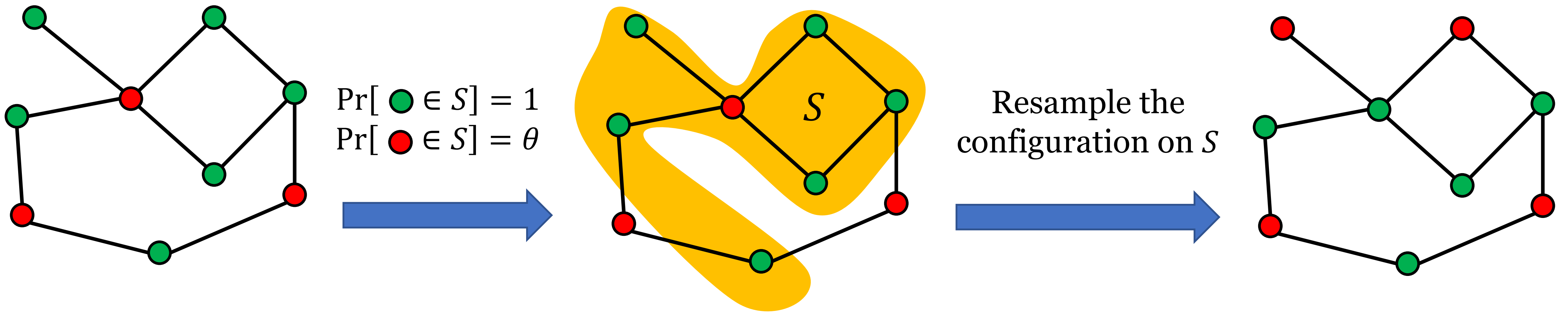
Start from an arbitrary feasible configuration $X \in \{-,+\}^V$

**For** each $t$ from 1 to $T$ **do**

**Down-Walk** • Construct $S \subseteq V$ be selecting each $v \in V$ independently with probability

$$p_v = \begin{cases} 1 & \text{if } X_v = - \\ \theta & \text{if } X_v = + \end{cases}$$

**Up-Walk** • Resample $X_S \sim \pi_S(\cdot \mid X_{V \setminus S})$ $\quad$ $\pi$: hardcore model $(G, \theta\lambda)$



## ***Field Dynamics***

**Input**: a **general distribution** $\mu$ over $\{-1,+1\}^V$, a parameter $\theta \in (0,1)$

Start from an arbitrary feasible configuration $X \in \{-,+\}^V$

**For** each $t$ from 1 to $T$ **do**

- Construct $S \subseteq V$ be selecting each $v \in V$ independently with probability

$$p_v = \begin{cases} 1 & \text{if } X_v = - \\ \theta & \text{if } X_v = + \end{cases}$$

- Resample $X_S \sim \pi_S(\cdot \mid X_{V \setminus S})$

$$\forall \sigma \in \{-,+\}^V, \qquad \pi(\sigma) \propto \mu(\sigma) \prod_{v \in V : \sigma_v = +} \theta$$

$\pi$: distribution $\mu$ with external field $\theta$

# ***Field Dynamics***

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- Resample $X_S \sim \pi_S(\cdot \mid X_{V\setminus S})$

**Proposition** (Field Dynamics): for any $\theta \in (0,1)$

Field dynamics has the unique stationary distribution $\mu$.

(irreducible, aperiodic and reversible)

# Spectral gap of Glauber dynamics

**Mixing lemma**

If $\lambda \le (1-\delta)\lambda_c(\Delta)$, for any $\theta \in (0,1)$

$$\mathrm{Gap}_{\mathrm{field}}(\lambda, \theta) \ge \theta^{O(1/\delta)}$$

**Comparison lemma**

For any $\lambda \ge 0$, for any $\theta \in (0,1)$

$$\mathrm{Gap}_{\mathrm{Glauber}}(\lambda) \ge \mathrm{Gap}_{\mathrm{field}}(\lambda, \theta) \cdot \mathrm{Gap}_{\mathrm{Glauber}}(\theta\lambda)$$

**Boosting theorem**

If $\lambda \le (1-\delta)\lambda_c(\Delta)$, for any $\theta \in (0,1)$

$$\mathrm{Gap}_{\mathrm{Glauber}}(\lambda) \ge \theta^{O(1/\delta)} \cdot \mathrm{Gap}_{\mathrm{Glauber}}(\theta\lambda)$$

Near-Critical Regime

Boosting with cost $O(1)$

Easy Regime

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**Comparison lemma**

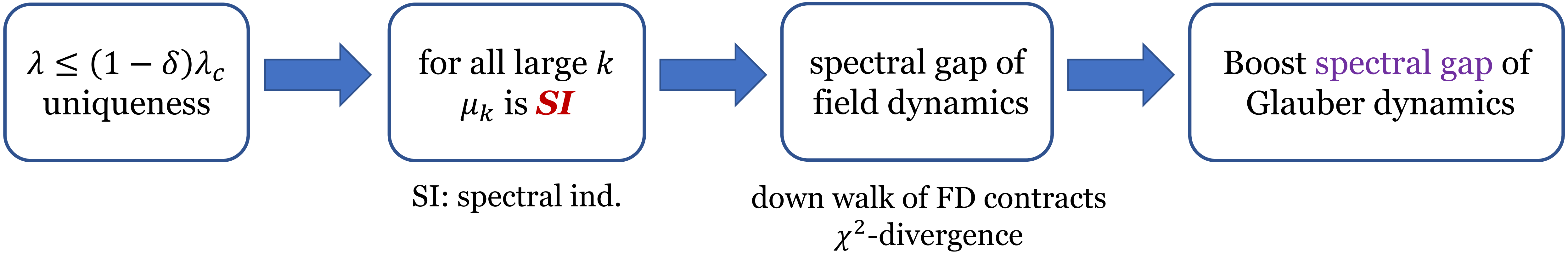
For any $\lambda \geq 0$, for any $\theta \in (0,1)$

$$\mathrm{Gap}_{\mathrm{Glauber}}(\lambda) \geq \mathrm{Gap}_{\mathrm{field}}(\lambda, \theta) \cdot \mathrm{Gap}_{\mathrm{Glauber}}(\theta\lambda)$$

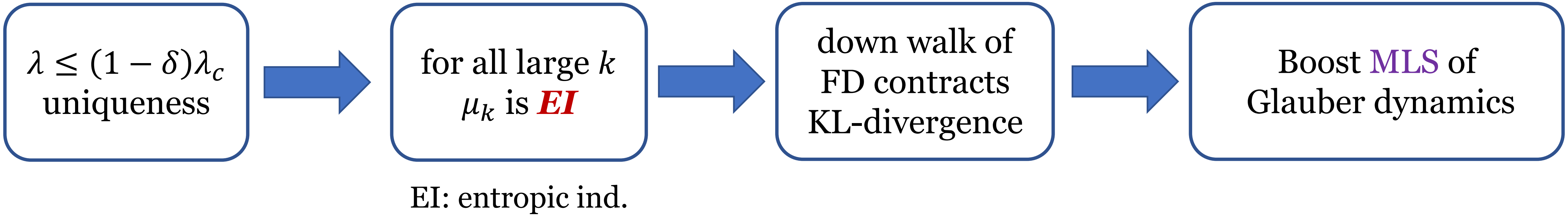
Proved by a calculation

$\theta = \frac{1}{10}$ → $\mathrm{Gap}_{\mathrm{Glauber}}(\theta\lambda) = \Omega\left(\frac{1}{n}\right)$, $\theta\lambda \in$ Dobrushin's regime → $\mathrm{Gap}_{\mathrm{Glauber}}(\lambda) = \Omega_\delta\left(\frac{1}{n}\right)$, $T_{\mathrm{mix}} = O_\delta(n^2 \log n)$

# Boosting spectral gap of Glauber dynamics




# Boosting modified log-Sobolev constant of Glauber dynamics




$$\mathrm{MLS}_{\mathrm{Glauber}}(\lambda) = \Omega_\delta\left(\frac{1}{n}\right) \text{ and } T_{\mathrm{mix}} = O_\delta(n \log n)$$

## Summary

- Hardcore model in the **uniqueness regime**
  - Optimal spectral gap and $O(n^2 \log n)$ mixing time
  - Optimal modified log-Sobolev constant and $O(n \log n)$ mixing time
- General distributions
  - Complete SI → boost spectral gap
  - Complete SI + marginal ratio bound → boost MLS constant
- A new Markov chain **field dynamics**

**Thank you!**

## Open problem

- More applications of field dynamics
  - Algorithmic applications (e.g., random cluster model [Chen and Zhang 2022])
- Extend our technique to ***general distributions*** beyond the Boolean domain i.e., $q$-coloring