Faster mixing of the Jerrum-Sinclair chain

Weiming Feng

The University of Hong Kong

Joint work with

Xiaoyu Chen, Zhe Ju, Tianshun Mao, Yitong Yin, Xinyuan Zhang

Nanjing University

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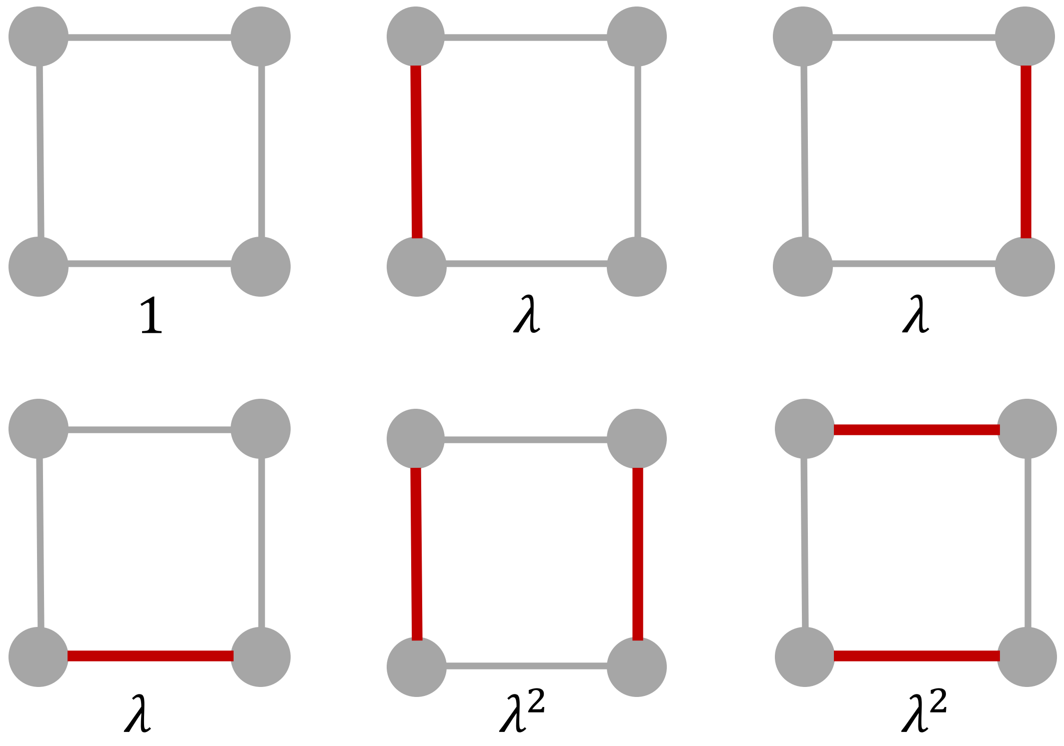
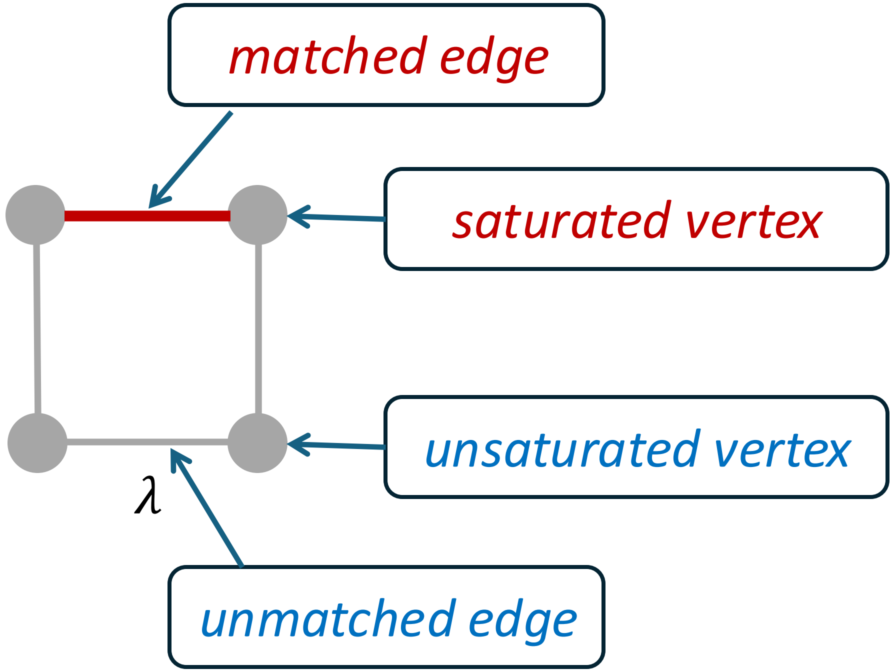
Monomer-dimer model

Simple graph $G = (V, E)$ and edge weight $\lambda > 0$

$$\forall \text{ mathing } M \subseteq E,\ \mu(M) \propto \lambda^{|M|}$$



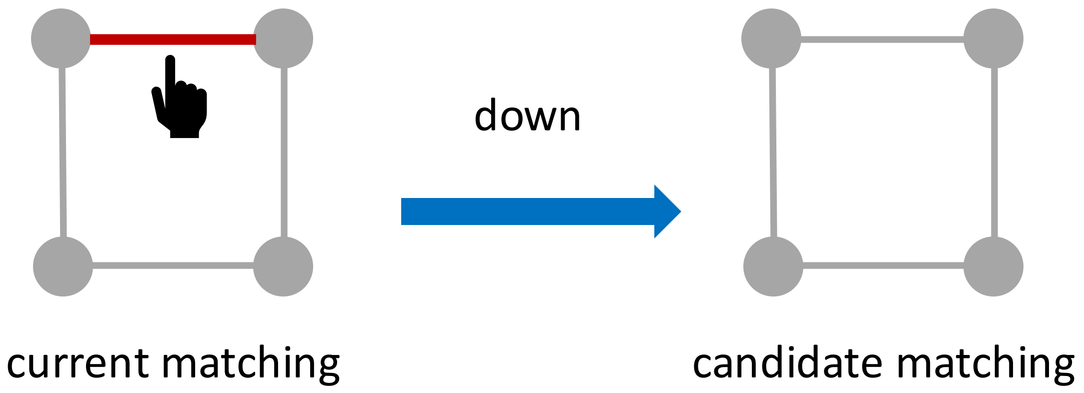


Jerrum-Sinclair chain (Metropolis chain)

Jerrum-Sinclair chain updates matching $X_t \to X_{t+1}$ by

- select an edge $e = \{u, v\} \in E$ u.a.r.
- propose a candidate matching M from X_t by

1) down transition*:* if $e \in X_t$, set $M \leftarrow X_t - e$



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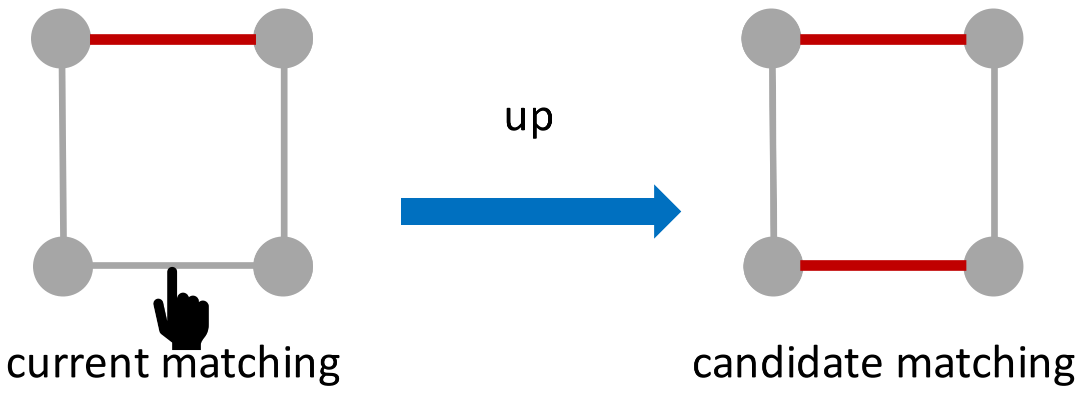
- propose a candidate matching M for X_{t+1} by
  1) down transition*:* if $e \in X_t$, set $M \leftarrow X_t - e$
  2) up transition*:* if both u, v are ***not saturated*** in X_t, set $M \leftarrow X_t + e$



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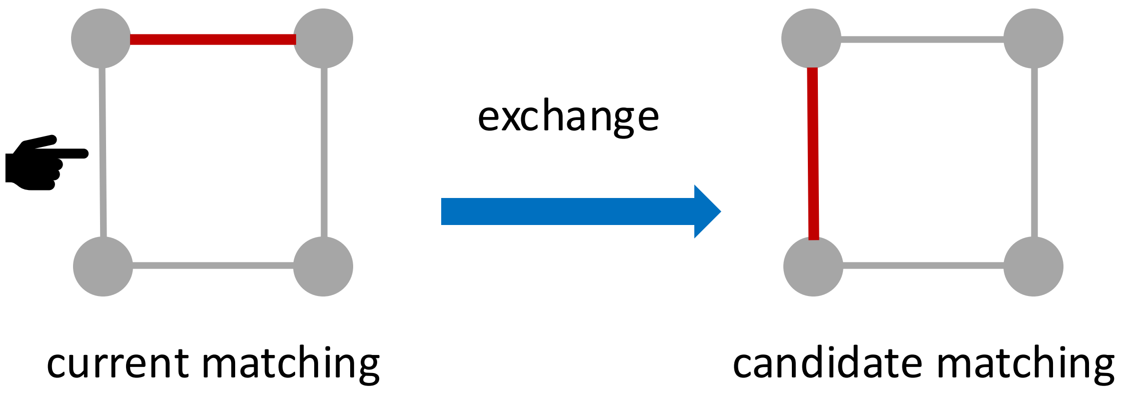
  3) exchange transition: if one endpoint is ***saturated*** and the other is ***not***, say
     u is saturated by edge f and v is not, set $M \leftarrow X_t + e - f$



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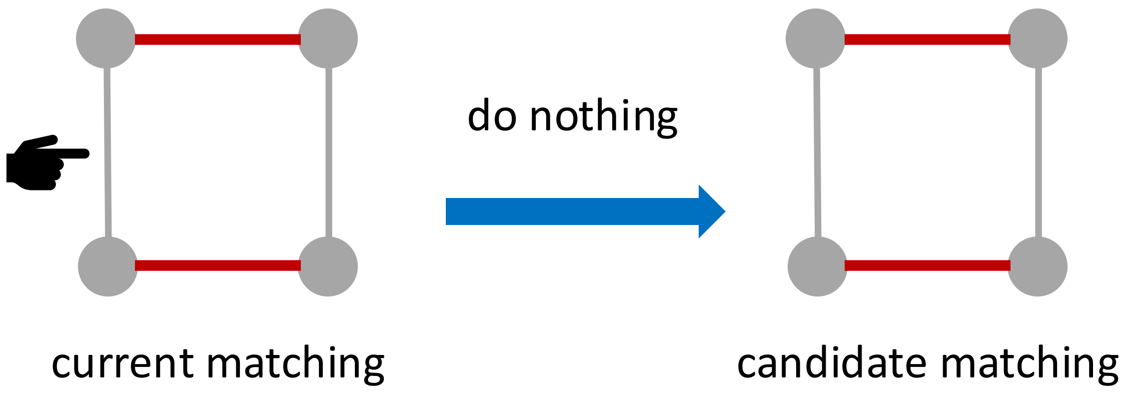
  4) otherwise (both u, v are saturated and $e \notin X_t$): set $M \leftarrow X_t$



Jerrum-Sinclair chain (Metropolis chain)

Jerrum-Sinclair chain P_{JS} updates matching $X_t \to X_{t+1}$ by

- select an edge $e = \{u, v\} \in E$ u.a.r.
- propose a candidate matching M for X_{t+1} by
  1) down transition: if $e \in X_t$, set $M \leftarrow X_t - e$
  2) up transition: if both u, v are ***not saturated*** in X_t, set $M \leftarrow X_t + e$
  3) exchange transition: if one endpoint is ***saturated*** and the other is ***not***, say u is saturated by edge f and v is not, set $M \leftarrow X_t + e - f$
  4) otherwise (both u, v are saturated and $e \notin X_t$): set $M \leftarrow X_t$
- with prob. $\min\left\{1, \frac{\mu(M)}{\mu(X_t)}\right\}$, accept M and set $X_{t+1} \leftarrow M$; otherwise, $X_{t+1} \leftarrow X_t$ (Metropolis filter)

1/2-Lazy Jerrum-Sinclair chain: $P_{JS\_zz} = \frac{1}{2}(P_{JS} + I)$

Mixing time: $T_{\mathrm{mix}}(P_{JS\_zz}) = \max_{X_0} \min\left\{ t > 0 \,\middle|\, \|X_t - \mu\|_{TV} \leq \frac{1}{4e} \right\}$

Mixing time results

Jerrum-Sinclair (1989): Graph with n vertices and m edges; constant $\lambda > 0$,

$$T_{\text{mix}}\left(P_{JS\_zz}\right) = \tilde{O}(mn^2)$$

Canonical Path: $O(nm)$-congestion with $O(n)$ path length ⟹ ***spectral gap*** $\gamma = \Omega\left(\frac{1}{nm}\right)$

Mixing time: $T_{\text{mix}}\left(P_{JS\_zz}\right) = O\left(\frac{1}{\gamma}\log\frac{1}{\mu_{\min}}\right)$, $\mu_{\min} = \min_{M} \mu(M)$ and $\log\frac{1}{\mu_{\min}} = \tilde{O}(n)$

Chen-Liu-Vigoda (2021): Graph with n vertices, m edges, and max degree Δ; constant $\lambda > 0$,

$$T_{\text{mix}}(\text{Glauber dynamics}) = O\left(\Delta^{\Delta^2} \cdot m \log n\right)$$

Spectral independence (local-to-global in HDX) ⟹ **modified log-Sobolev const.** $\alpha = \Omega_{\Delta}\left(\frac{1}{m}\right)$

Our Result: $T_{\text{mix}}\left(P_{JS\_zz}\right) = O(\Delta m \cdot \min\{n, \Delta \log \Delta \log n\}) = \tilde{O}(m\Delta^2)$

- **Spectral gap:** $\Omega\left(\frac{1}{m\Delta}\right)$
- **Log-Sobolev const.:** $\Omega\left(\frac{1}{m\Delta^2}\right)$

Corollary: $T_{\text{mix}}(\text{Glauber dynamics}) = \tilde{O}(\Delta^3 \cdot m)$

General results

- ***Distribution*** μ over $\Omega \subseteq [q]^E$ for finite domain $[q] = \{1,2,\dots,q\}$ and variable set E
- ***Random variable*** $F = f(X)$ for a function $f: \Omega \to \mathbb{R}$ and $X \sim \mu$

Variance: $\mathrm{Var}[F] = \mathrm{Var}_\mu[f] = \mathbb{E}[F^2] - \mathbb{E}[F]^2$

Entropy: $\mathrm{Ent}[F] = \mathrm{Ent}_\mu[f] = \mathbb{E}[F \log F] - \mathbb{E}[F] \log \mathbb{E}[F]$

- ***Reversible Markov chain*** Q for μ: $\forall x, y, \mu(x)Q(x,y) = \mu(y)Q(y,x)$

Dirichlet form: $\mathcal{E}_Q(f,f) = \frac{1}{2}\sum_{xy\in\Omega} \mu(x)Q(x,y)\big(f(x) - f(y)\big)^2$ for all $f: \Omega \to \mathbb{R}$

Poincáre Inequality (Spectral Gap)

$$\gamma(Q) \cdot \mathrm{Var}_\mu[f] \le \mathcal{E}_Q(f,f)$$

$$\Rightarrow T_{\mathrm{mix}}\left(\frac{Q+I}{2}\right) = O\left(\frac{1}{\gamma(Q)} \log \frac{1}{\mu_{\min}}\right)$$

Log-Sobolev Inequality

$$\rho(Q) \cdot \mathrm{Ent}_\mu[F^2] \le \mathcal{E}_Q(f,f)$$

$$\Rightarrow T_{\mathrm{mix}}(Q) = O\left(\frac{1}{\rho(Q)} \log\log \frac{1}{\mu_{\min}}\right)$$

Family of Markov chains

- For a subset $\Lambda \subseteq E$, a pinning $\tau \in [q]^{E \setminus \Lambda}$ outside Λ, define conditional distribution

$$\mu^{\tau} = (\text{distribution of } X \sim \mu \text{ conditional on } X_{E \setminus \Lambda} = \tau)$$

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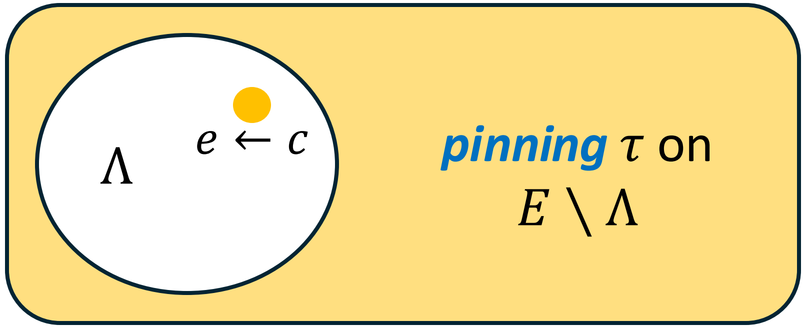
- Markov chain Q^τ is a reversible chain for μ^τ

a family of chains $\mathcal{Q} = \{\, Q^\tau \mid \tau \text{ is a pinning} \,\}$

Example: $\mathcal{Q}$ is a family of *Glauber dynamics* or a family of *Metropolis chains*.

Concave Dirichlet forms: $\forall \Lambda \subseteq E, \forall \tau \in [q]^{E\setminus\Lambda}$, the Markov chain Q^τ for μ^τ satisfies

$$\frac{1}{|\Lambda|}\sum_{e\in\Lambda} \mathbb{E}_{c\sim\mu_e^\tau}\left[\mathcal{E}_{Q^{\tau\wedge(e\leftarrow c)}}(f,f)\right] \leq \mathcal{E}_{Q^\tau}(f,f).$$



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$$\underbrace{\frac{1}{|\Lambda|}\sum_{e\in\Lambda} \mathbb{E}_{c\sim\mu_e^\tau}\left[\mathcal{E}_{Q^{\tau\wedge(e\leftarrow c)}}(f,f)\right]}_{\text{average of Dirichlet forms}} \leq \underbrace{\mathcal{E}_{Q^\tau}(f,f)}_{\text{Dirichlet forms of } Q^\tau \text{ for } \mu^\tau}.$$

Markov chain for conditional distributions
with ***one more pinned variable***

Local functional inequalities

Distribution μ over $\Omega \subseteq [q]^E$, random variables $F = f(X)$, where $f\colon \Omega \to \mathbb{R}$ and $X \sim \mu$

α-local Poincáre Inequality

$$\alpha \cdot \underbrace{\sum_{e \in E}}_{\text{sum over all variables in } E} \mathrm{Var}\left[\underbrace{\mathbb{E}[\, F \mid X_e \,]}_{\text{a random variable in } \mathbb{R} \text{ support size at most } q}\right] \le \mathcal{E}_Q(f, f)$$

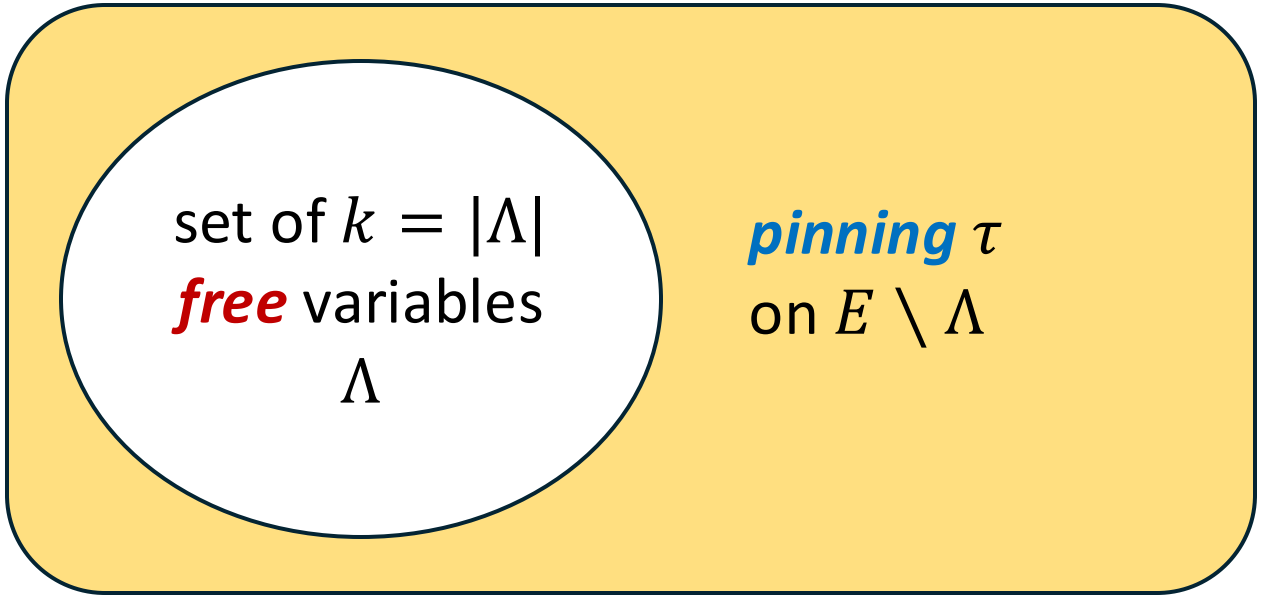
α-local log-Sobolev Inequality

$$\alpha \cdot \sum_{e \in E} \mathrm{Ent}\left[\mathbb{E}[\, F^2 \mid X_e \,]\right] \le \mathcal{E}_Q(f, f)$$

Local functional inequalities for a family of Markov chains Q

$(\alpha_1, \alpha_2 \ldots, \alpha_{|E|})$-local Poincáre Inequality

$\forall \Lambda \subseteq E, \forall \tau \in [q]^{E \setminus \Lambda}$, the Markov chain Q^τ for μ^τ satisfies the $\alpha_{|\Lambda|}$-local Poincáre Inequality



Local functional inequalities for a family of Markov chains $\mathcal{Q}$

$(\alpha_1, \alpha_2 \ldots, \alpha_{|E|})$-local Poincáre Inequality

$\forall \Lambda \subseteq E, \forall \tau \in [q]^{E \setminus \Lambda}$, the Markov chain Q^τ for μ^τ satisfies the $\alpha_{|\Lambda|}$-local Poincáre Inequality

$$\alpha_{|\Lambda|} \cdot \sum_{e \in E} \mathrm{Var}\left[\mathbb{E}[\, F \mid X_e \,]\right] \leq \mathcal{E}_{Q^\tau}(f, f),$$

where $F = f(X)$ and $X \sim \mu^\tau$

Local functional inequalities for a family of Markov chains Q

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$\forall \Lambda \subseteq E, \forall \tau \in [q]^{E \setminus \Lambda}$, the Markov chain Q^τ for μ^τ satisfies the $\alpha_{|\Lambda|}$-local Poincáre inequality

$(\alpha_1, \alpha_2 \ldots, \alpha_{|E|})$-local log-Sobolev Inequality

$\forall \Lambda \subseteq E, \forall \tau \in [q]^{E \setminus \Lambda}$, the Markov chain Q^τ for μ^τ satisfies the $\alpha_{|\Lambda|}$-local log-Sobolev inequality

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Local-to-global theorem for functional inequalities

For a family of Markov chains Q with the concave Dirichlet forms

$(\alpha_1, \alpha_2 \dots, \alpha_{|E|})$-local Poincáre Inequality $\Rightarrow$ Poincáre constant $\gamma(Q) \geq \left(\sum_{k=1}^{|E|} \frac{1}{k\alpha_k}\right)^{-1}$

$(\alpha_1, \alpha_2 \dots, \alpha_{|E|})$-local log-Sobolev Inequality $\Rightarrow$ Log-Sobolev constant $\rho(Q) \geq \left(\sum_{k=1}^{|E|} \frac{1}{k\alpha_k}\right)^{-1}$

Q: Markov chain for $\mu = \mu^\emptyset$ without pinning

How to establish local functional inequalities?

α-local Poincáre Inequality

$$\alpha \cdot \sum_{e \in E} \mathrm{Var}\left[\mathbb{E}[\, F \mid X_e \,]\right] \leq \mathcal{E}_Q(f,f)$$

α-local log-Sobolev Inequality

$$\alpha \cdot \sum_{e \in E} \mathrm{Ent}\left[\mathbb{E}[\, F^2 \mid X_e \,]\right] \leq \mathcal{E}_Q(f,f)$$

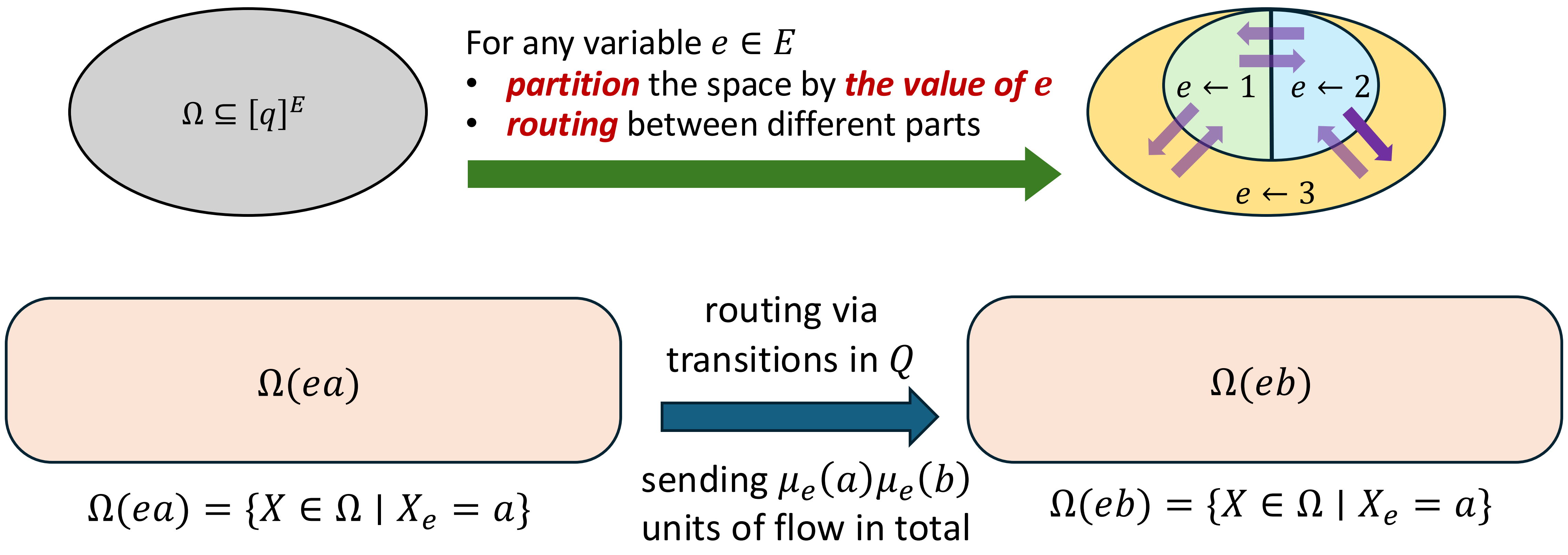
Transport Flow

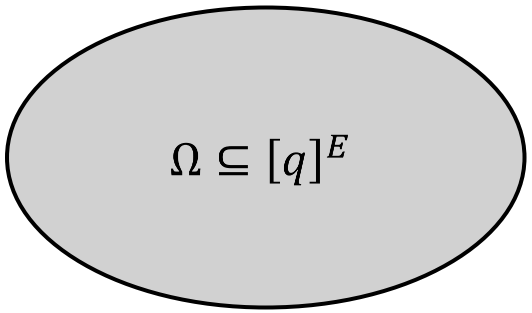
Given a Markov chain Q, a transport flow Γ from a distribution ν to a distribution π is ***a distribution of paths*** such that $\gamma = (x_0, x_1, \dots, x_\ell) \sim \Gamma$ satisfies

- The starting point $s(\gamma) = x_0 \sim \nu$
- The endpoint $t(\gamma) = x_\ell \sim \pi$

$\big(s(\gamma), t(\gamma)\big)$ forms a ***coupling*** of ν and π

- Every pair of adjacent points (x_i, x_{i+1}) is a ***transition*** in Q


$\Omega \subseteq [q]^E$
For any variable $e \in E$
• ***partition*** the space by ***the value of e***
• ***routing*** between different parts
$e \leftarrow 1$
$e \leftarrow 2$
$e \leftarrow 3$
$\Omega(ea)$
routing via transitions in Q
$\Omega(eb)$
$\Omega(ea) = \{X \in \Omega \mid X_e = a\}$
sending $\mu_e(a)\mu_e(b)$ units of flow in total
$\Omega(eb) = \{X \in \Omega \mid X_e = a\}$



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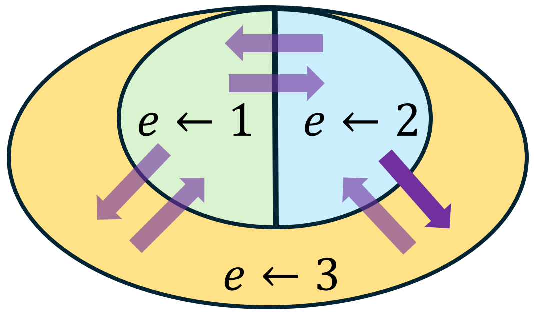


$\forall \sigma \in \Omega(ea)$ needs to ***send***
$\mu_e(a)\mu_e(b)\mu^{e\leftarrow a}(\sigma)$ unit of flow

$\Omega(ea) = \{X \in \Omega \mid X_e = a\}$

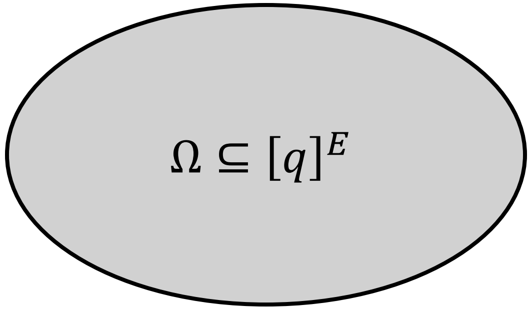
routing via
transitions in Q

sending $\mu_e(a)\mu_e(b)$
units of flow in total

$\forall \tau \in \Omega(eb)$ needs to ***receive***
$\mu_e(a)\mu_e(b)\mu^{e\leftarrow b}(\tau)$ unit of flow

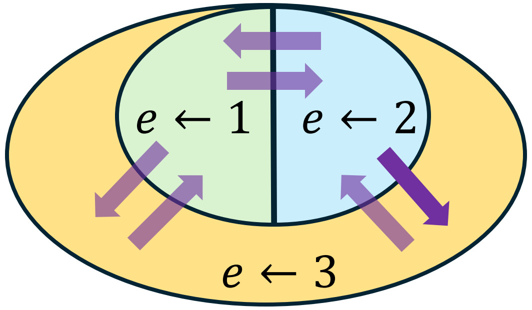
$\Omega(eb) = \{X \in \Omega \mid X_e = a\}$

Sending $\mu_e(a)\mu_e(b)$ units of flow via a ***random path***
from the ***transport flow*** $\Gamma_e^{a\to b}$ from $\mu^{e\leftarrow a}$ to $\mu^{e\leftarrow b}$



For any variable $e \in E$

- ***partition*** the space by ***the value of e***
- ***routing*** between different parts



$\forall \sigma \in \Omega(ea)$ needs to ***send***
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$\Omega(ea) = \{X \in \Omega \mid X_e = a\}$

routing via transitions in Q

sending $\mu_e(a)\mu_e(b)$ units of flow in total

$\forall \tau \in \Omega(eb)$ needs to ***receive***
$\mu_e(a)\mu_e(b)\mu^{e\leftarrow b}(\tau)$ unit of flow

$\Omega(eb) = \{X \in \Omega \mid X_e = a\}$

Every path $\gamma = (x_0, x_1, \dots, x_\ell)$

send $\mu_e(a)\mu_e(b) \Pr_{X \sim \Gamma_e^{a\to b}}[X = \gamma]$ units of flow from x_0 to x_ℓ

Local Poincáre inequality via transport flow

If there exists a ***family of transport flow***

$$\{\Gamma_e^{a\to b} \text{ from } \mu^{e\leftarrow a} \text{ to } \mu^{e\leftarrow b} \mid e \in E, a, b \in [q]\}$$

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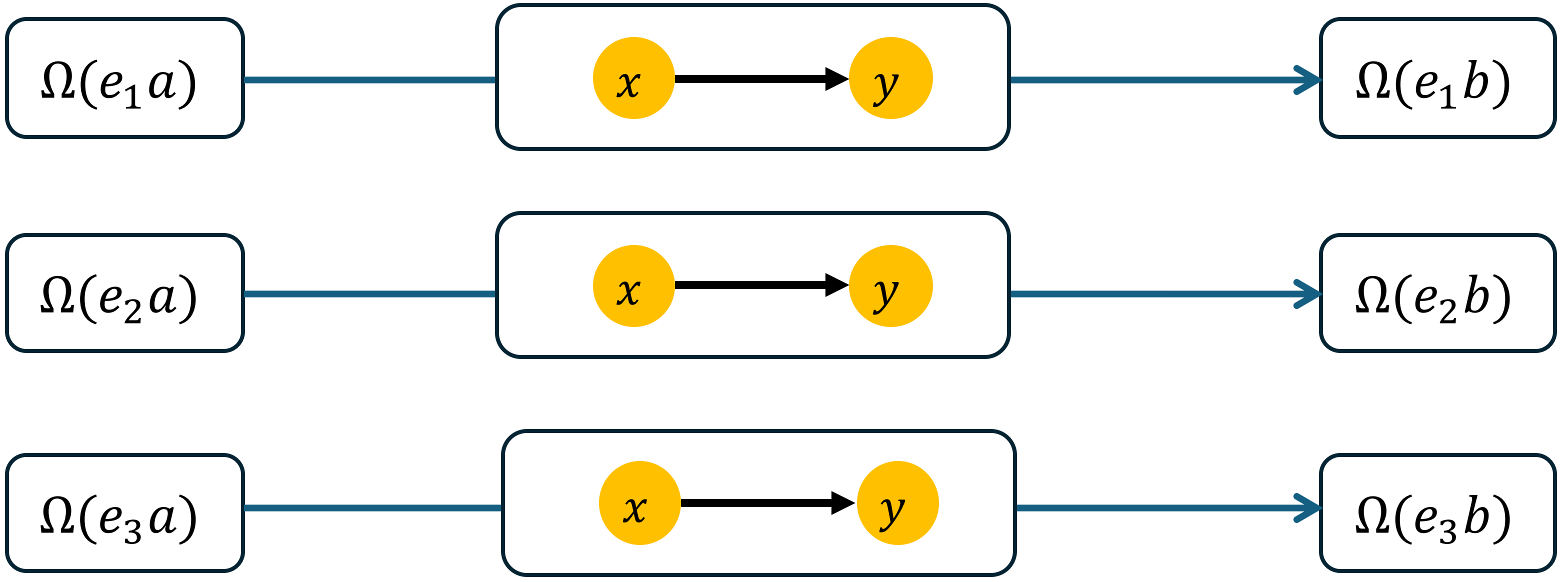
- (κ-***expected congestion***) For ***any transition*** $(x \to y)$ in Q, and any $a, b \in [q]$,

$$\sum_{e\in E} \mu_e(a)\mu_e(b) \cdot \Pr_{\gamma\sim\Gamma_e^{a\to b}}[(x \to y) \in \gamma] \le \boldsymbol{\kappa} \cdot \mu(x)Q(x,y).$$



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- (L-***expected length***) For any $e \in E$, any $a, b \in [q]$,

$$\mathbb{E}_{\gamma\sim\Gamma_e^{a\to b}}[\ell(\gamma)] \le \boldsymbol{L}$$

Slightly different definitions *are used in the paper to improve the application*

α-local Poincáre Inequality

$$\Rightarrow \quad \alpha \cdot \sum_{e\in E} \mathrm{Var}\left[\mathbb{E}[\,F \mid X_e\,]\right] \le \mathcal{E}_Q(f,f) \text{ with } \alpha = \Omega\left(\frac{1}{q^2\kappa L}\right)$$

Local Poincáre inequality via transport flow

Family of transport flow
- low expected congestion
- low expected length

→ local Poincáre Inequality

Construct transport flow $\Gamma_{\mathrm{e}}^{a\to b}$ from $\mu^{e\leftarrow a}$ to $\mu^{e\leftarrow b}$ such that for $\gamma = (x_0, x_1, \dots, x_\ell) \sim \Gamma$

- The starting point $s(\gamma) = x_0 \sim \mu^{e\leftarrow a}$
- The endpoint $t(\gamma) = x_\ell \sim \mu^{e\leftarrow b}$

$\big(s(\gamma), t(\gamma)\big)$ forms a ***coupling*** of $\mu^{e\leftarrow a}$ and $\mu^{e\leftarrow a}$

- Every pair of adjacent points (x_i, x_{i+1}) is a ***transition*** in Q

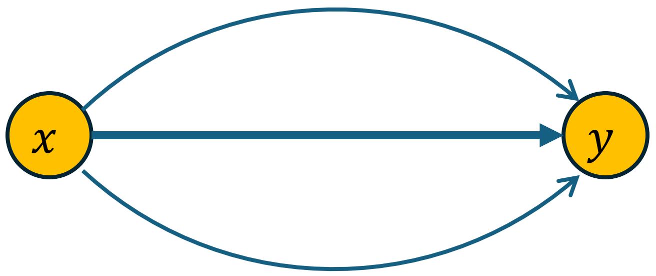
Find a ***good coupling*** with small ***expected discrepancy*** between $\mu^{e\leftarrow a}$ and $\mu^{e\leftarrow b}$

Canonical path and multicommodity flow

The technique [Diaconis and Stroock 91] [Sinclair 92] is to bound ***global variance***

$$\alpha \cdot \mathrm{Var}[F] \leq \mathcal{E}_Q(f, f)$$

For ***any pair*** $(x, y) \in \Omega$



Send $\mu(x)\mu(y)$ units of flow

$\Leftrightarrow$

A ***transport flow*** from

Dirac distribution ν (with $\nu(x) = 1$)

to

Dirac distribution π (with $\pi(y) = 1$)

- ***Canonical path***: sending flow through one path
- ***Multicommodity flow***: sending flow through a distribution of paths

Proof outline: Local Poincáre inequality via transport flow

$$\mathrm{Var}\big[\mathbb{E}[\,F \mid X_e\,]\big] = \frac{1}{2}\sum_{a,b\in[q]} \mu_e(a)\mu_e(b)\big(\mathbb{E}[F \mid X_e = a] - \mathbb{E}[F \mid X_e = b]\big)^2$$

local variance

amount of flow

Proof outline: Local Poincáre inequality via transport flow

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By definition $F = f(x)$

$$= \frac{1}{2}\sum_{a,b\in[q]} \mu_e(a)\mu_e(b)\left(\mathbb{E}_{X\sim\mu^{e\leftarrow a}}[f(X)] - \mathbb{E}_{Y\sim\mu^{e\leftarrow b}}[f(Y)]\right)^2$$

beginning of the transport flow

ending of the transport flow

Proof outline: Local Poincáre inequality via transport flow

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Coupling : $(x_0, x_\ell) \sim \left(\mu^{e\leftarrow a}, \mu^{e\leftarrow b}\right)$

$$= \frac{1}{2}\sum_{a,b\in[q]} \mu_e(a)\mu_e(b)\left(\mathbb{E}_{\gamma=(x_0,x_1,\ldots x_\ell)\sim\Gamma_e^{a\to b}}[f(x_0) - f(x_\ell)]\right)^2$$

sample a random path from the flow

Proof outline: Local Poincáre inequality via transport flow

$$\mathrm{Var}\big[\mathbb{E}[\,F \mid X_e\,]\big] = \frac{1}{2}\sum_{a,b\in[q]} \mu_e(a)\mu_e(b)(\mathbb{E}[F \mid X_e = a] - \mathbb{E}[F \mid X_e = b])^2$$

By definition $F = f(x)$
$$= \frac{1}{2}\sum_{a,b\in[q]} \mu_e(a)\mu_e(b)\left(\mathbb{E}_{X\sim\mu^{e\leftarrow a}}[f(X)] - \mathbb{E}_{Y\sim\mu^{e\leftarrow b}}[f(Y)]\right)^2$$

Coupling : $(x_0, x_\ell) \sim (\mu^{e\leftarrow a}, \mu^{e\leftarrow b})$
$$= \frac{1}{2}\sum_{a,b\in[q]} \mu_e(a)\mu_e(b)\left(\mathbb{E}_{\gamma=(x_0,x_1,\ldots x_\ell)\sim\Gamma_e^{a\to b}}[f(x_0) - f(x_\ell)]\right)^2$$

Telescoping sum along the path
$$= \frac{1}{2}\sum_{a,b\in[q]} \mu_e(a)\mu_e(b)\left(\mathbb{E}_{\gamma=(x_0,x_1,\ldots x_\ell)\sim\Gamma_e^{a\to b}}\left[\sum_{1\le i\le\ell}\big(f(x_i) - f(x_{i-1})\big)\right]\right)^2$$

Proof outline: Local Poincáre inequality via transport flow

$$\mathrm{Var}\big[\mathbb{E}[\,F \mid X_e\,]\big] = \frac{1}{2}\sum_{a,b\in[q]} \mu_e(a)\mu_e(b)(\mathbb{E}[F \mid X_e = a] - \mathbb{E}[F \mid X_e = b])^2$$

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Summing by enumerating transitions

$$= \frac{1}{2}\sum_{a,b\in[q]} \mu_e(a)\mu_e(b)\left(\mathbb{E}_{\gamma\sim\Gamma_e^{a\to b}}\left[\sum_{(x\to y)\in Q}\big(f(x) - f(y)\big)\mathbf{1}[(x\to y)\in\gamma]\right]\right)^2$$

Proof outline: Local Poincáre inequality via transport flow

$$\mathrm{Var}\big[\mathbb{E}[\,F \mid X_e\,]\big] = \frac{1}{2}\sum_{a,b\in[q]} \mu_e(a)\mu_e(b)(\mathbb{E}[F \mid X_e = a] - \mathbb{E}[F \mid X_e = b])^2$$

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- using **Cauchy–Schwarz inequality** on the term
- the rest of the proof follows from the standard analysis in [Sinclair 92]

Local log-Sobolev inequality via transport flow

If there exists a ***family of transport flow***

$$\{\Gamma_e^{a \to b} \text{ from } \mu^{e \leftarrow a} \text{ to } \mu^{e \leftarrow b} \mid e \in E, a, b \in [q]\}$$

- (κ-***strong expected congestion***) For ***any transition*** $(x \to y)$ in Q, and any $a, b \in [q]$,

$$\sum_{e \in E} \mu_e(a)\mu_e(b) \cdot \mathbb{E}_{\gamma \sim \Gamma_e^{a \to b}}\left[\ell(\gamma) \cdot \mathbf{1}[(x \to y) \in \gamma]\right] \le \boldsymbol{\kappa} \cdot \mu(x)Q(x, y).$$

add the length of the path into the expectation

Local log-Sobolev inequality via transport flow

If there exists a ***family of transport flow***

$$\{\Gamma_e^{a\to b} \text{ from } \mu^{e\leftarrow a} \text{ to } \mu^{e\leftarrow b} \mid e \in E, a, b \in [q]\}$$

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$\Longrightarrow$

α-local log-Sobolev Inequality

$$\alpha \cdot \sum_{e\in E} \mathrm{Ent}\left[\mathbb{E}[\, F^2 \mid X_e\,]\right] \le \mathcal{E}_Q(f,f) \text{ with } \alpha = \Omega\left(\frac{1}{q^2\kappa\log\frac{1}{\phi}}\right)$$

$\phi = \min\{\mu_e(c) \mid e \in E, c \in [q]\}$ is the ***marginal lower bound***

Proof outline: log-Sobolev inequality via transport flow

marginal lower bound of X_e → marginal lower bound of $\mathbb{E}[F^2 \mid X_e]$ → $\mathrm{Ent}\left[\mathbb{E}[F^2 \mid X_e]\right] \le \dfrac{\log\left(\frac{1}{\phi} - 1\right)}{1 - 2\phi} \mathrm{Var}\left[\sqrt{\mathbb{E}[F^2 \mid X_e]}\right]$

$$\mathrm{Var}\left[\sqrt{\mathbb{E}[F^2 \mid X_e]}\right] \le \sum_{a,b \in [q]} \mu_e(a)\mu_e(b) \cdot \mathbb{E}_{\gamma=(x_0,\ldots,x_\ell)\sim\Gamma_e^{a\to b}}\left[\left(\sum_{1\le i\le \ell} f(x_i) - f(x_{i-1})\right)^2\right]$$

by convexity of $h(x,y) = \left(\sqrt{x} - \sqrt{y}\right)^2$

Using **Cauchy–Schwarz inequality** on the term

Application to Jerrum-Sinclair chain

Transport flow for ***JS-chain family***
- low expected congestion
- low expected length

→ ***local*** functional inequalities →

- Poincáre inequality
- log-Sobolev inequality

⇒ mixing

For μ^τ with pinnings $\tau \in \{0,1\}^{E-\Lambda}$, free variables in Λ, the Jerrum-Sinclair chain Q^τ: $X_t \to X_{t+1}$
- Pick an edge $e \in \Lambda$ uniformly at random
- Construct a candidate matching M from X_t
- Accept or reject M via Metropolis filter w.r.t. μ^τ

The family of Jerrum-Sinclair chains $Q = \{\, Q^\tau \mid \tau \,\}$ satisfies
- $(\alpha_1, \alpha_2, \dots, \alpha_{|E|})$-***local Poincáre inequality*** with $\alpha_k = \Omega_\lambda\left(\frac{1}{k\Delta}\right)$
- $(\alpha_1, \alpha_2, \dots, \alpha_{|E|})$-***log Sobolev inequality*** with $\alpha_k = \Omega_\lambda\left(\frac{1}{k\Delta^2 \log \Delta}\right)$

Proved by ***transport flow***

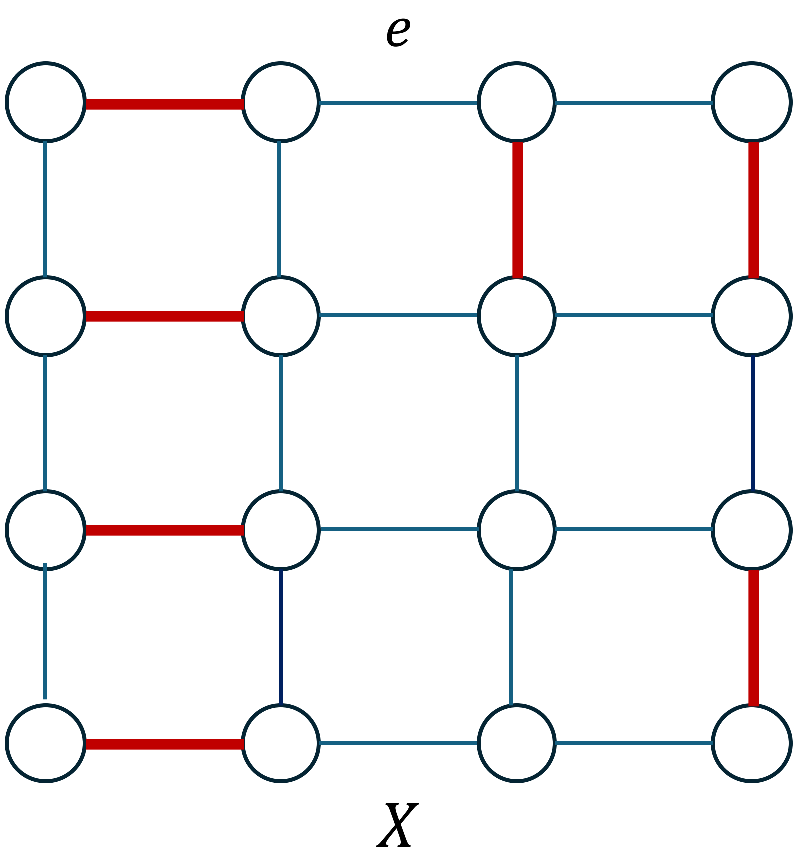
Fix an edge $e \in E$, construct transport flow from $\mu^{e\leftarrow\mathrm{unmatched}}$ to $\mu^{e\leftarrow\mathrm{matched}}$

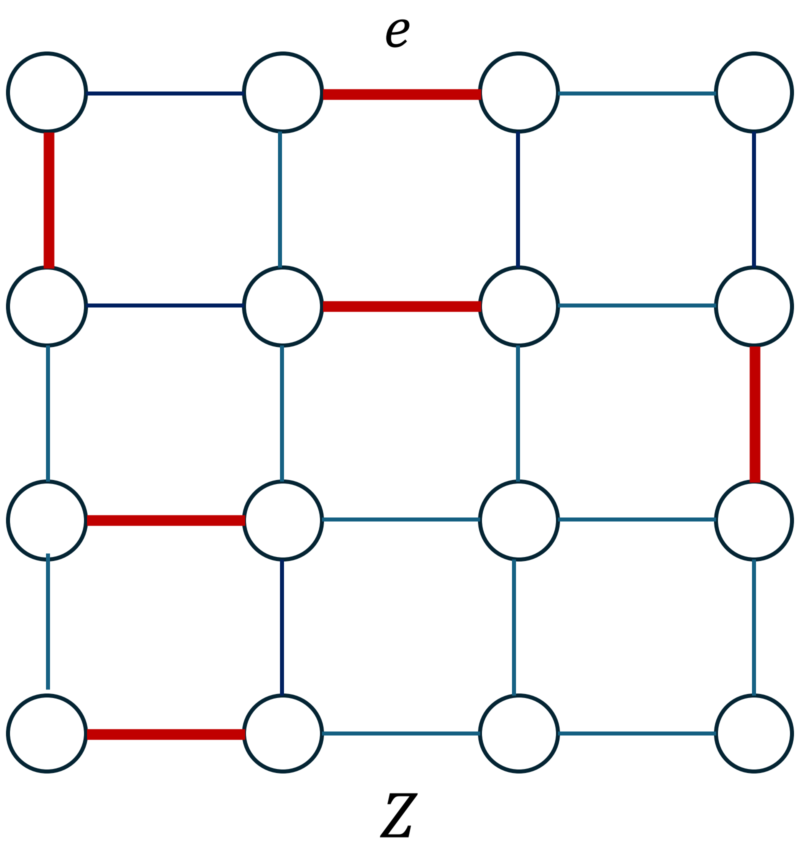
- Sample (X, Y) from the ***local-flipping coupling*** of $\mu^{e\leftarrow\mathrm{unmatched}}$ to $\mu^{e\leftarrow\mathrm{matched}}$
- Construct canonical path from X to Y using Jerrum and Sinclair's construction

Local flipping coupling → **Low discrepancy** between X and Y → **Low congestion** and **low path length**

Local Flipping Coupling

- Sample $X \sim \mu^{e\leftarrow\text{unmatched}}$ and $Z \sim \mu^{e\leftarrow\text{matched}}$ independently
- The difference between X and Y are paths and cycles, find the unique one B containing e
- Let $Y = Z_B \cup X_{E-B}$ (flipping B in X to obtain Y)





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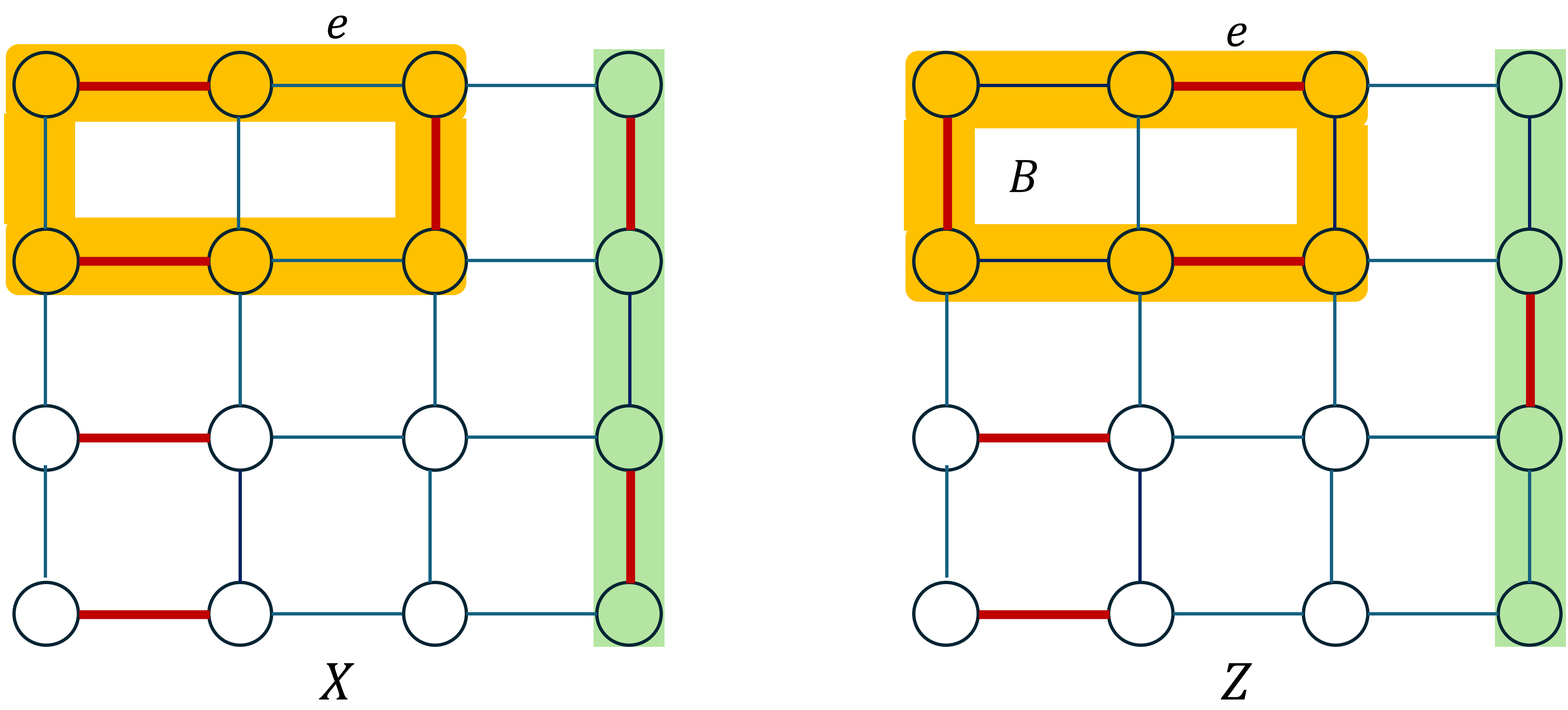

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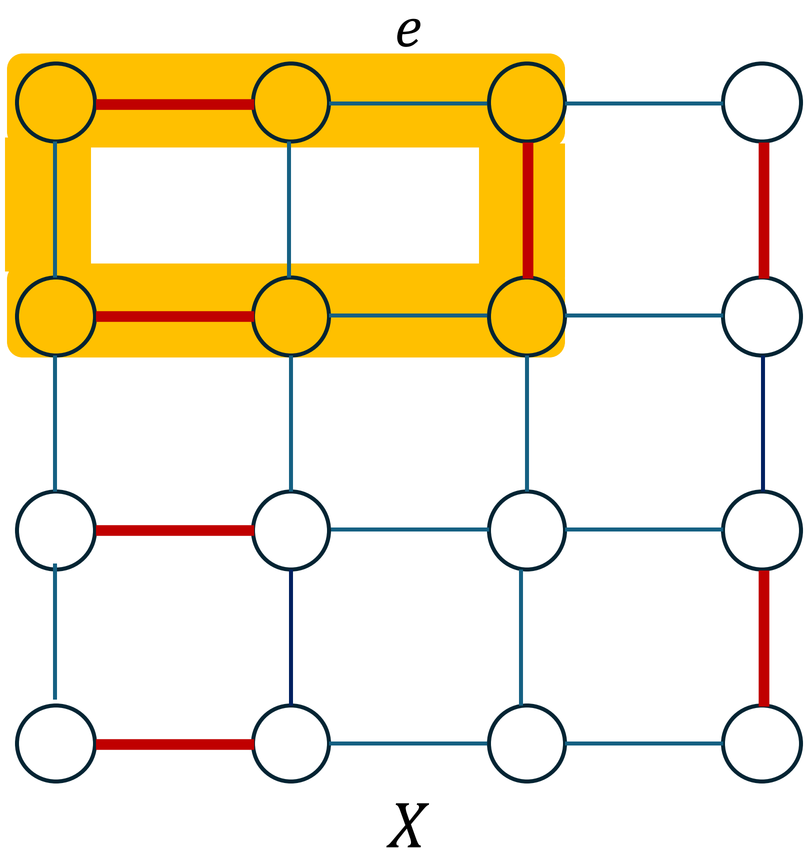

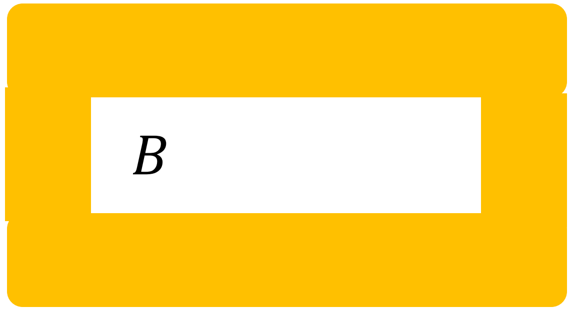

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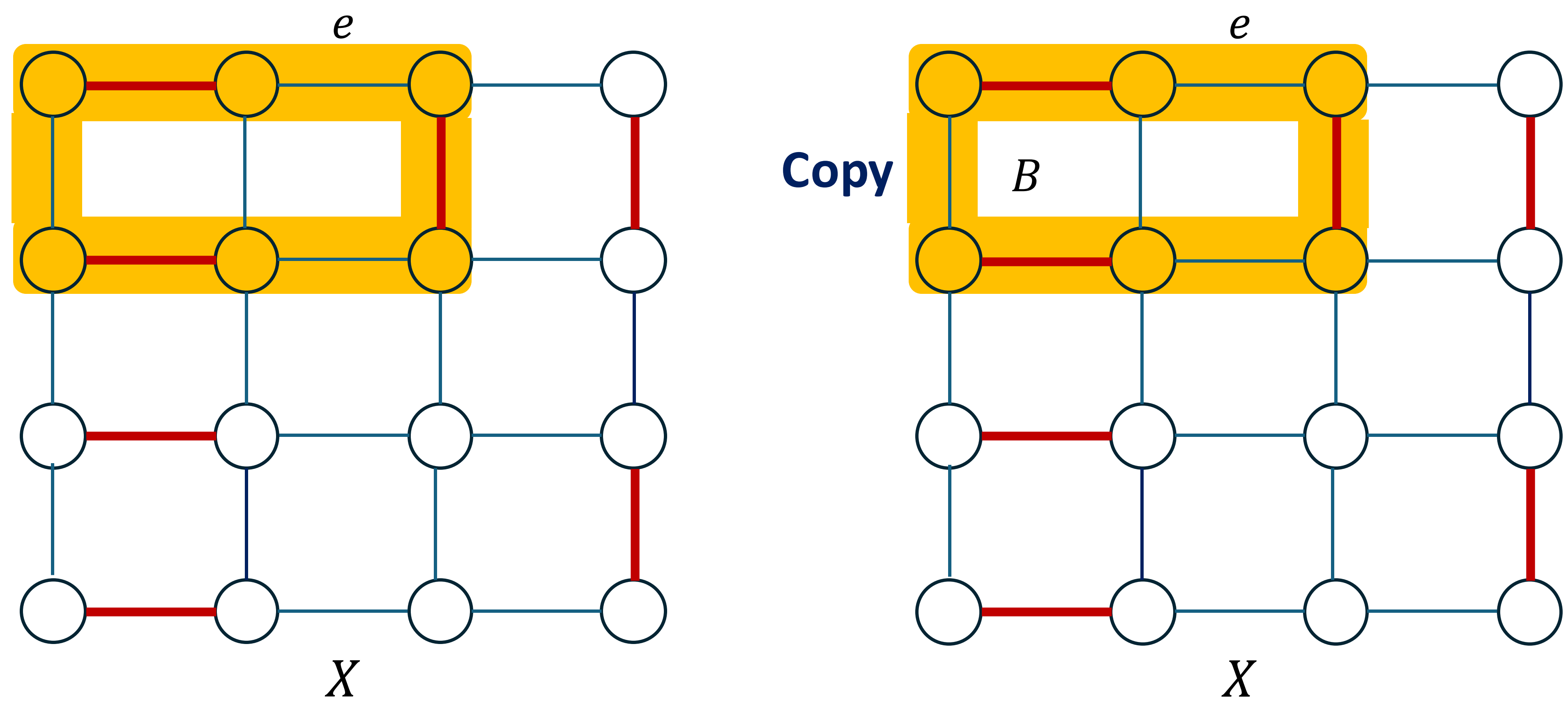

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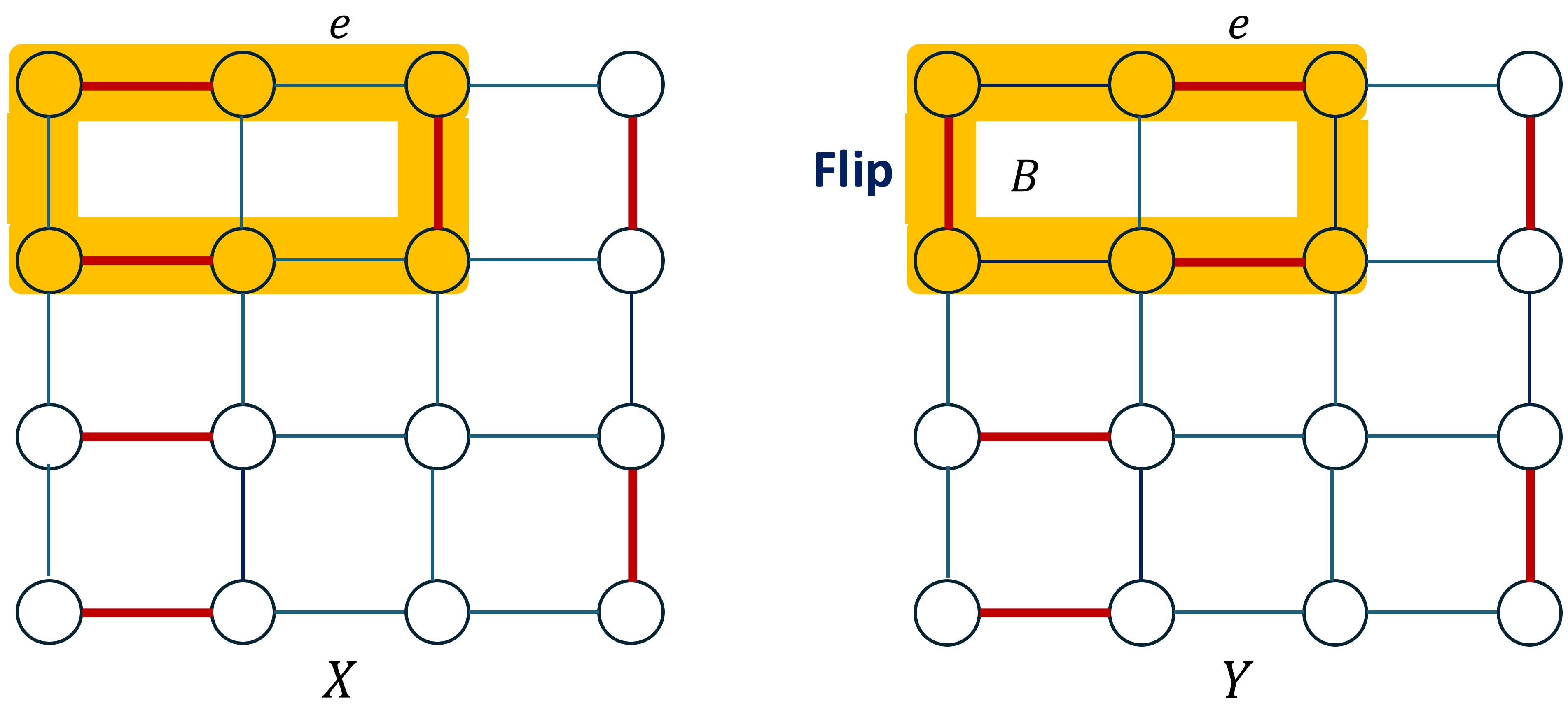

Transport Flow

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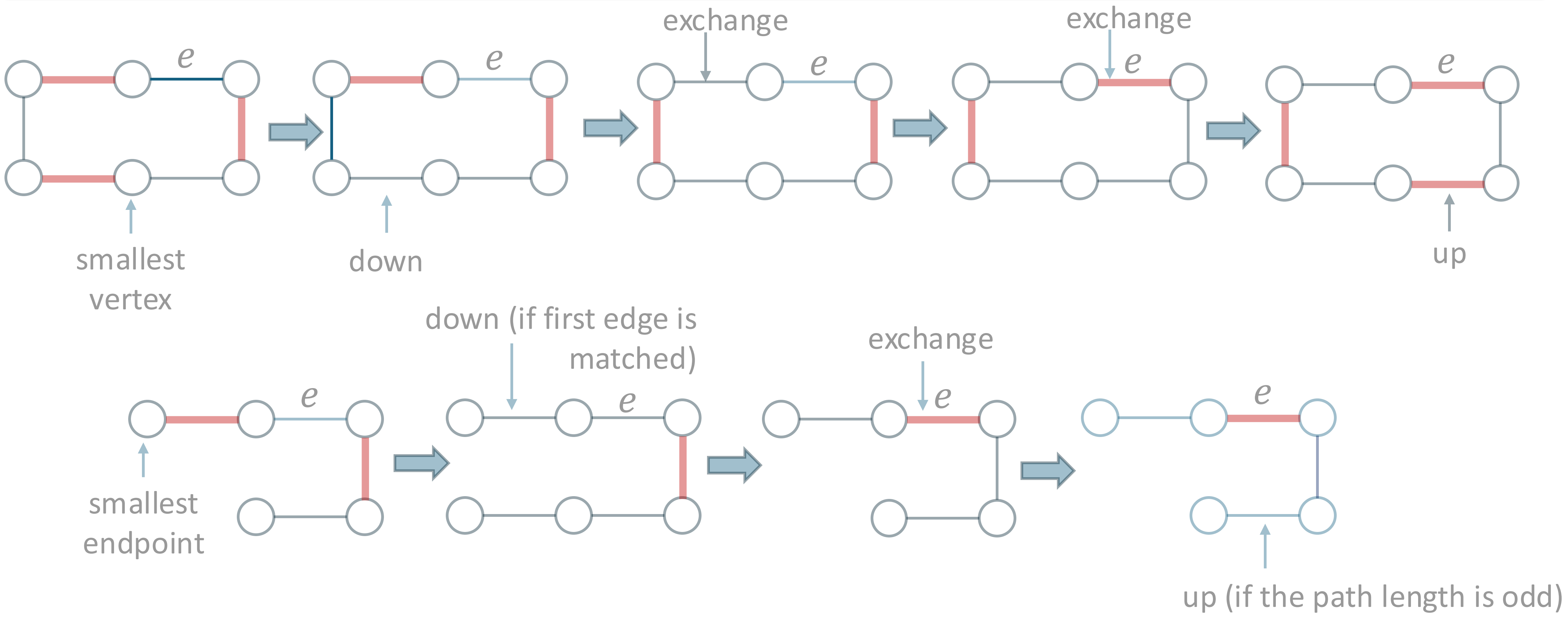
- Construct canonical path from X to Y using Jerrum and Sinclair's construction



Proof overview of expected length and congestion

- Analyze coupling via ***local reviewing process***
- Disagreement percolation

For $(X, Y) \sim \mathcal{C}_e$ from local flipping coupling

- Length bound $\mathbb{E}[|X \oplus Y|] \leq O_\lambda(\sqrt{\Delta})$
- One sided bound $\mathbb{E}[|X \oplus Y| \mid X = x] \leq O_\lambda(\Delta)$

Expected congestion and strong congestion analysis

$$\sum_{e \in E} \mu_e(a)\mu_e(b) \cdot \mathbb{E}_{\gamma \sim \Gamma_e}\left[\mathbf{1}[(x \to y) \in \gamma]\right] \leq \boldsymbol{\kappa} \cdot \mu(x)Q(x,y).$$

$$\sum_{e \in E} \mu_e(a)\mu_e(b) \cdot \mathbb{E}_{\gamma \sim \Gamma_e}\left[\ell(\gamma) \cdot \mathbf{1}[(x \to y) \in \gamma]\right] \leq \boldsymbol{\kappa} \cdot \mu(x)Q(x,y).$$

sum of $|E|$ different couplings

$\gamma \sim \Gamma_e$

- Sample starting and ending points from coupling
- Construct the path deterministically

The randomness is only from locally flipping coupling

Proof overview of expected length and congestion

- Analyze coupling via ***local reviewing process***
- Disagreement percolation

For $(X,Y) \sim \mathcal{C}_e$ from local flipping coupling

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$$\sum_{e\in E} \mu_e(a)\mu_e(b) \cdot \mathbb{E}_{\gamma\sim\Gamma_e}[\ell(\gamma) \cdot \mathbf{1}[(x \to y) \in \gamma]] \le \boldsymbol{\kappa} \cdot \mu(x)Q(x,y).$$

sum of $|E|$ different couplings

decoupling lemma

analyze ***one coupling*** with a different function inside the expectation

bound congestion by constructing ***injection***

(guided by Jerrum-Sinclair's analysis)

Open problems

- Lower discrepancy coupling of $\mu^{e\leftarrow a}$ and $\mu^{e\leftarrow b}$
- Construction of canonical paths

$\Rightarrow$

- Poincáre inequality
- log-Sobolev inequality

- Sharp bound for Jerrum-Sinclair chain: $\tilde{O}(m\sqrt{\Delta})$ mixing?
- More applications?

  Improving the mixing bound for e.g. the permanent, the Ising model,

  the switch/flip chain for sampling regular graphs…