# Rapid mixing from spectral independence beyond the Boolean domain


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# Glauber dynamics

## Sampling from joint distribution

| | |
|---|---|
| **Set of variables** | $V$ |
| **Finite domain** | $[q] = \{1,2,\dots,q\}$ for $q \geq 2$ |
| **Joint distribution** | $\mu$ over $\Omega = \mathrm{supp}(\mu) \subseteq [q]^V$ |
| **Problem** | draw random samples from $\mu$ |

## Fundamental MCMC: Glauber dynamics

Start from an arbitrary feasible configuration $X \in \Omega$;

**For** each $t$ from 1 to $T$ **do**

- pick a variable $v \in V$ uniformly at random;
- resample $X_v \sim \mu_v(\cdot \mid X_{V\setminus\{v\}})$;

**Return** $\boldsymbol{X}$;

# Example: proper $q$-coloring

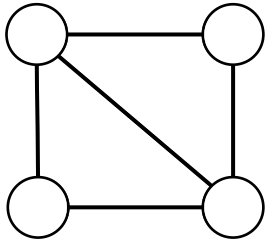
## **Uniform proper $q$-coloring**

**Undirected graph** $G = (V, E)$

**Finite set of colors** $[q] = \{1,2, \dots, q\}$

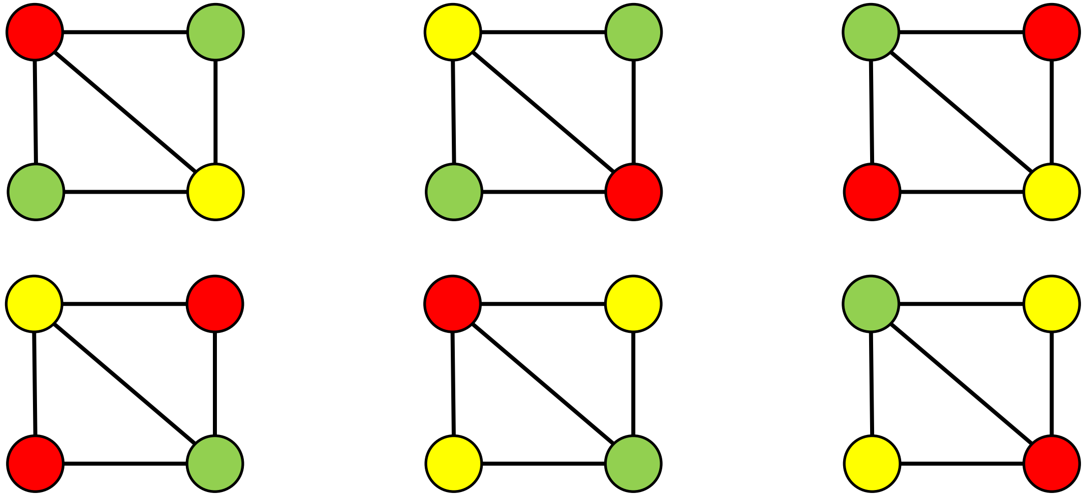
**Gibbs distribution** $\mu$ **uniform distribution** over $\Omega$

$$\Omega = \{X \in [q]^V \mid X \text{ is a proper coloring}\}$$



graph $G = (V, E)$

colors $[q] = \{$  $\}$



$\Omega$: set of all proper colors

# Example: proper $q$-coloring

## Uniform proper $q$-coloring

**Undirected graph** $G = (V, E)$

**Finite set of colors** $[q] = \{1,2, \ldots, q\}$

**Gibbs distribution** $\mu$ **uniform distribution** over $\Omega = \{X \in [q]^V \mid X \text{ is a proper coloring}\}$

**Problem** sample proper coloring u.a.r.

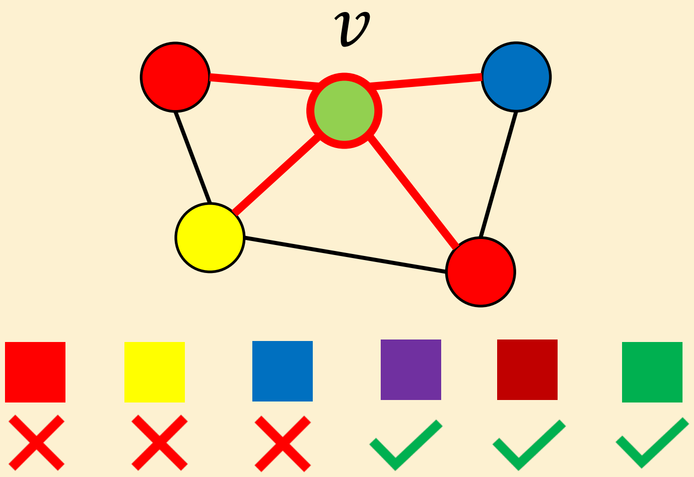
## Glauber dynamics for proper $q$-coloring

Start from an arbitrary proper coloring $X \in \Omega$;

**For** each $t$ from 1 to $T$ **do**

- pick a vertex $v \in V$ uniformly at random;
- resample $X_v$ from $[q] \setminus \{X_u \mid u \in \Gamma(v)\}$ uniformly at random;

**Return** $X$;



# Convergence

**Glauber dynamics:** Markov chain over $\Omega$

**Transition Matrix** $P \in \mathbb{R}^{\Omega \times \Omega}$

**Glauber dynamics is *reversible***

**detailed balance equation** with respect to $\mu$

$$\forall X, Y \in \Omega, \mu(X)P(X,Y) = \mu(Y)P(Y,X)$$

→ *Stationary distribution* $\mu P = \mu$

move among any states with positive probability

**Proposition (convergence)**

If Glauber dynamics is **connected**, it converges to **unique** stationary distribution $\mu$.

If $\boldsymbol{q \geq \Delta + 2}$, Glauber dynamics converges to uniform distribution over $q$-colorings.

# Mixing time

***How fast*** does the Glauber dynamics converge to stationary distribution $\mu$?

Glauber dynamics $X_0, X_1, X_2, \ldots$ where each $X_i \in \Omega \subseteq [q]^V$

**Mixing time**

$$T_{\text{mix}} = \max_{X_0 \in \Omega} \min \left\{ t \;\middle|\; d_{TV}(X_t, \mu) \le \frac{1}{4e} \right\},$$

$d_{TV}(X_t, \mu)$: the **total variation distance** between $X_t$ and $\mu$.

The Glauber dynamics is **rapid mixing** if

$$T_{\text{mix}} = \text{Poly}(n) \qquad n = |V| = \#\{\text{variables}\}$$

✓ Sample from an exponential space $|\Omega| = \exp(O(n))$ within polynomial steps $T_{\text{mix}} = \text{poly}(n)$.

# Open problems

Under what **condition** of the distribution $\mu$

the Glauber dynamics for $\mu$ rapid mixing ?

Under what **relation** between $q$ and max degree $\Delta$

the Glauber dynamics for coloring rapid mixing ?

# Previous works

**Glauber dynamics for graph coloring**

General graphs [Jer95,Vig00, SS97, CDMPP19]

current best result $q \geq \left(\frac{11}{6} - \epsilon_0\right)\Delta$ [CDMPP19]

Special graphs [DF01, Hay03,HV03,GMP05, HV06, Mol04, Hay13,DFHV13]

**High-dimensional expansion (HDX)**

Strongly log-concave distribution [ALOV19, CGM19]

**Spectral independence** with **Boolean domain $\{0,1\}^V$** [ALO20]

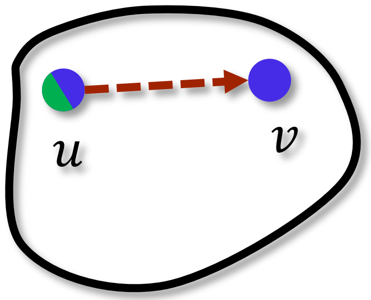
**Spectral Independence** → Mixing up to uniqueness for **hardcore model** [ALO20], **anti-ferro 2-spin systems** [CLV20]

# Our results

- A **spectral independence** condition for **general distribution.**
- **Rapid mixing** of Glauber dynamics from spectral independence.
  - combinatorial proof: **coupling;**
  - algebraic proof for Boolean variables [ALO20].
- Application: a new rapid mixing regime for **graph coloring.**
  - relate spectral independence with **correlation decay;**
  - a refined **recursive coupling** [GMP05] argument.

# Our results

**Result (I).** **A spectral independence condition beyond the Boolean domain.**

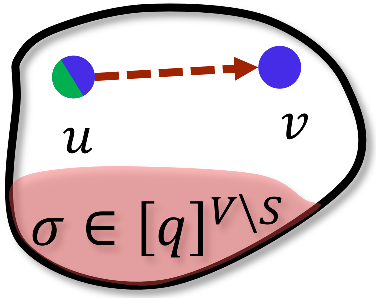



**maximum influence** on $v$ caused by a **disagreement** on $u$

$\mu$: a distribution over $\Omega \subseteq [q]^V$

$|V|\times|V|$ **influence matrix** $\Psi \in \mathbb{R}^{V\times V}$ with $\Psi(u,u)=0$ and

$$\Psi(u,v) = \max_{i,j\in[q]} d_{TV}\left(\mu_v(\cdot \mid u \leftarrow i), \mu_v(\cdot \mid u \leftarrow j)\right)$$




**influence matrix** for **conditional distribution**

For any subset $S \subseteq V$, any feasible $\sigma \in [q]^{V\setminus S}$

$\mu_S^\sigma$ distribution on $S$ conditional on $\sigma$

**influence matrix** $\Psi_S^\sigma \in \mathbb{R}^{S\times S}$ for **conditional distribution**

$$\Psi_S^\sigma(u,v) = \max_{i,j\in[q]} d_{TV}\left(\mu_v^\sigma(\cdot \mid u \leftarrow i), \mu_v^\sigma(\cdot \mid u \leftarrow j)\right)$$

# Our results

**Result (I).** **A spectral independence condition beyond the Boolean domain**

**Spectral independence [This work]**

**There is a constant $\boldsymbol{C} > 0$ such that**

**for all conditional distributions $\boldsymbol{\mu_S^\sigma}$,**

***spectral radius of influence matrices*** $\rho(\Psi_S^\sigma) \leq \boldsymbol{C}$.

**Spectral independence for Boolean variables**[Anari, Liu, Oveis Gharan 20]

Distribution over **Boolean domain** $\{0,1\}^V$

signed influence matrix: $I_S^\sigma(u,v) = \mu_v^\sigma(1 \mid u \leftarrow 1) - \mu_v^\sigma(1 \mid u \leftarrow 0)$.

**Relation:** $\Psi_S^\sigma(u,v) = |I_S^\sigma(u,v)|$.

**Spectral independence:** for all influence matrices, max eigenvalue $\lambda_{\max}(I_S^\sigma) \leq C$.

# Our results

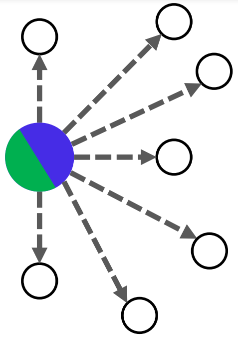
**Result (II).** **Rapid mixing** of Glauber dynamics from **spectral independence**

**Theorem [This work]**

$\mu$ is spectrally independent with constant $C$

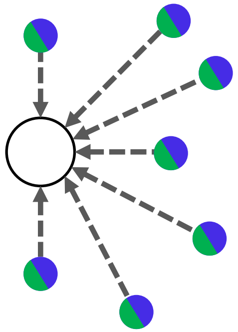
$$\Rightarrow T_{\text{mix}} = O\left(n^{1+2C} \log\left(\frac{1}{\mu_{\min}}\right)\right),$$

where $\mu_{\min} = \min_{X \in \Omega} \mu(X)$.



**Bounded one-to-all influence**

$$\text{spectral radius} \le \sum_{v \in S} \Psi_S^{\sigma}(u, v) \le C$$



**Bounded all-to-one influence**

$$\text{spectral radius} \le \sum_{u \in S} \Psi_S^{\sigma}(u, v) \le C$$

Spectral Independence → Rapid Mixing

# Our results

**Result (III).** Rapid mixing for $q$-coloring on triangle-free graph with $q > 1.763\Delta$

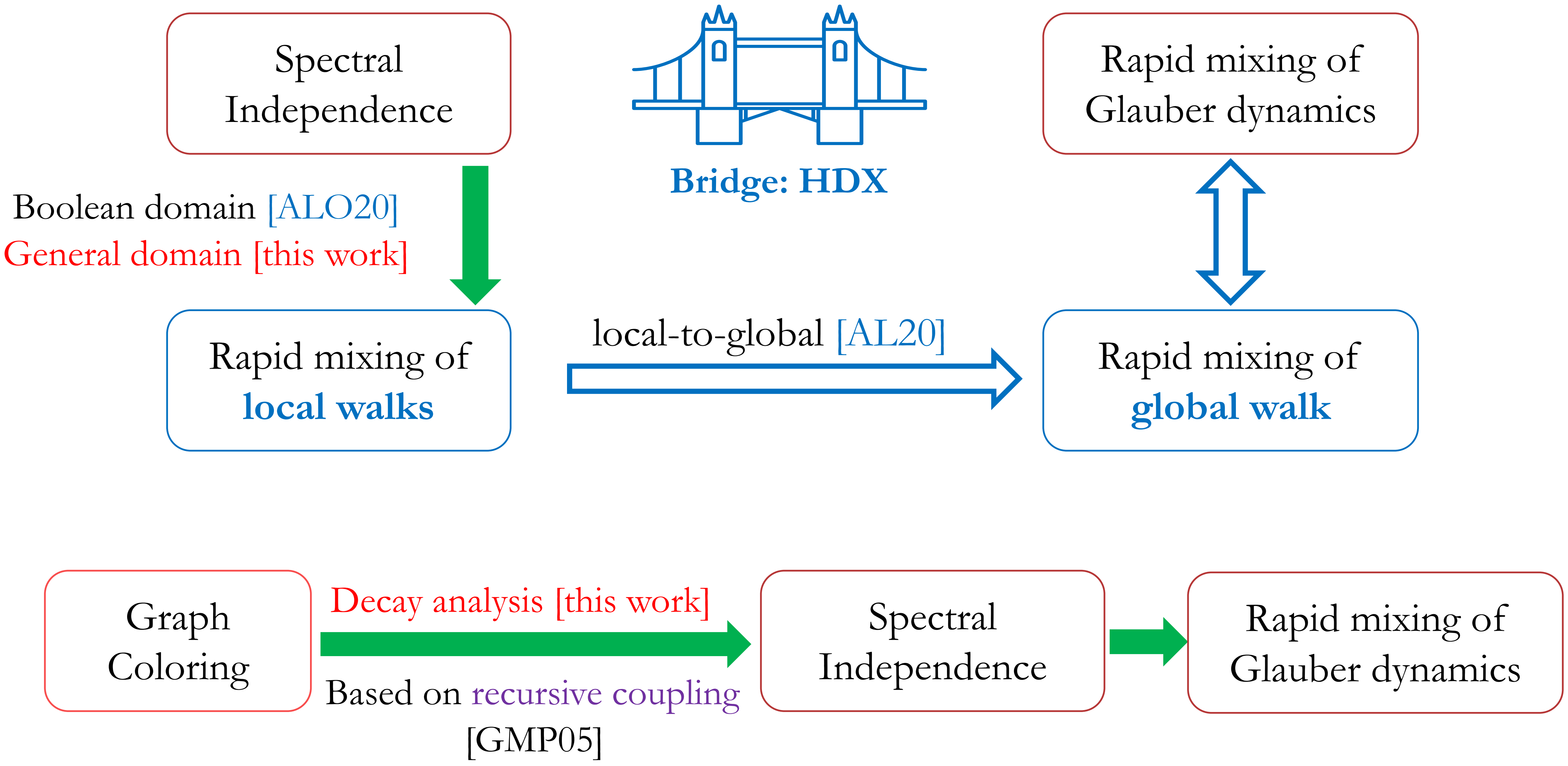
**Theorem [This work]** ← constant

Triangle free graph and $q \geq (\alpha + \delta)\Delta$ where $\alpha \approx 1.763$ s.t. $\alpha = \exp(1/\alpha)$

$$\Rightarrow T_{\text{mix}} \leq n^{2+O(1/\delta)} \log q .$$

| Work | Regime | Girth | Addition condition | Mixing time |
|---|---|---|---|---|
| [GMP05] | $q > \alpha\Delta$ | $\geq 4$ | $\Delta = O(1)$ + amenable graph | $O(n^2)$ |
| [HV06] | $q \geq (\alpha + \delta)\Delta$ | $\geq 4$ | $\Delta = \Omega(\log n)$ | $O(n \log n)$ |
| [DFHV13] | $q \geq (\alpha + \delta)\Delta$ | $\geq 5$ | $\Delta \geq \Delta_0(\delta)$ | $O(n \log n)$ |
| This work | $q \geq (\alpha + \delta)\Delta$ | $\geq 4$ | -- | $n^{2+O(1/\delta)} \log q$ |

# Proof outline

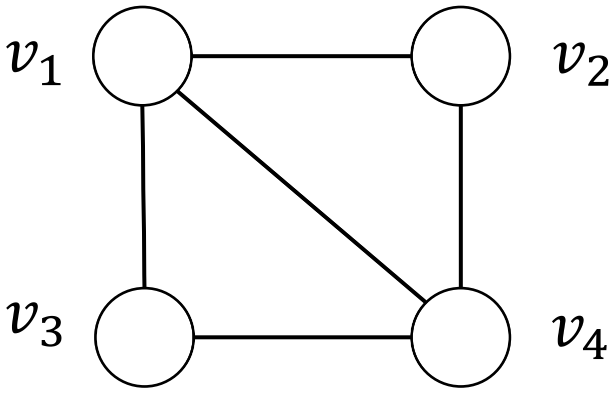


# Lazy local random walk

**State space** $U = \{ (v, i) \mid v \in V, i \in [q] \}$

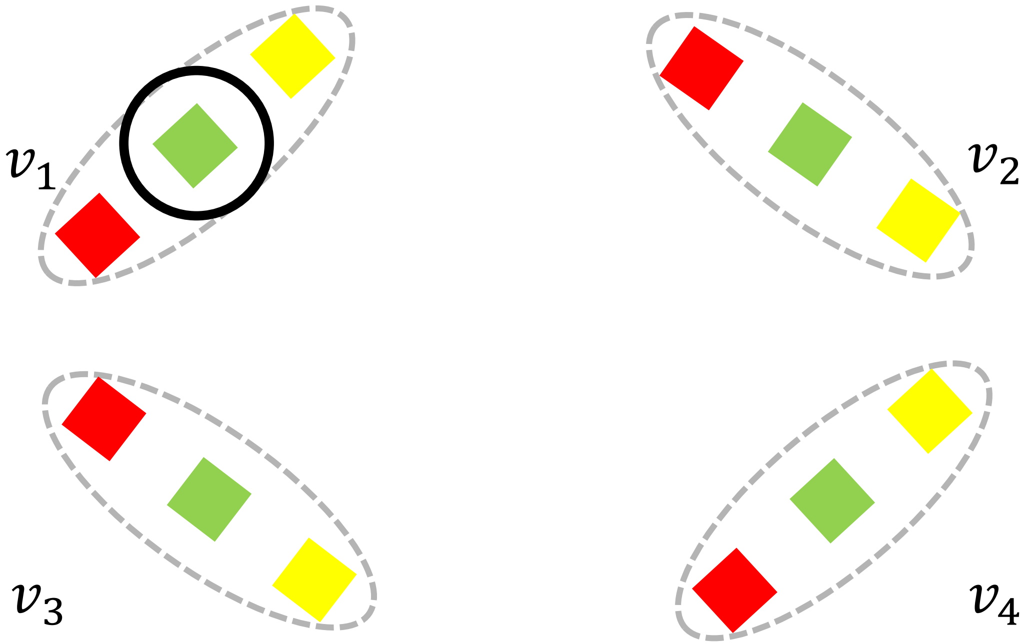
Current state $(v, i) \in U$. Transition $(v, i) \to (u, j)$

- pick a vertex $u \in V$ uniformly at random;
- sample $j \sim \mu_u(\cdot \mid v \leftarrow i)$.




graph $G = (V, E)$

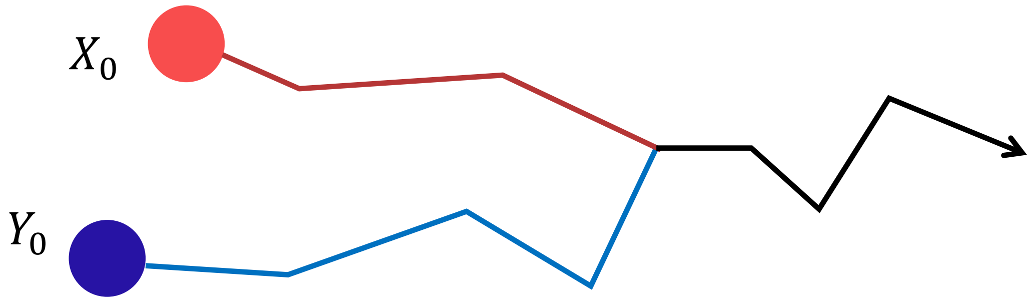
colors $[q] = \{$ ■ ■ ■ $\}$

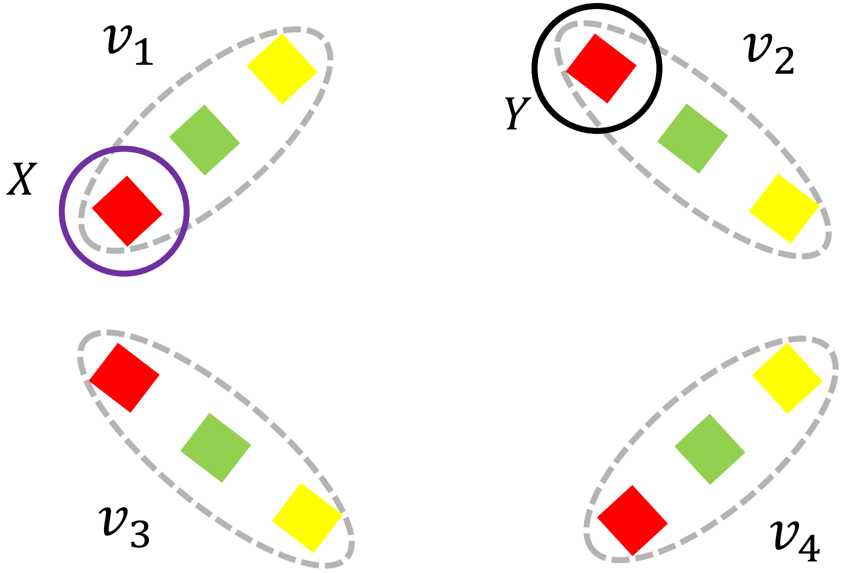


# Our technique: coupling

**Coupling $(\boldsymbol{X}_t, \boldsymbol{Y}_t)_{t\geq 0}$ of local walk**

- start from two states $X_0, Y_0 \in U$
- two chains $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ follow local walk




**Coupling**

**Current state** $X_t = (u, i)$ and $Y_t = (v, j)$

**Next state** $X_{t+1} = (u', i')$ and $Y_{t+1} = (v', j')$

- Pick the **same** $u' = v' \in V$ uniformly at random;
- Sample $(i', j')$ from the **optimal coupling** between $\mu_{u'}(\cdot \mid u \leftarrow i)$ and $\mu_{v'}(\cdot \mid v \leftarrow j)$.

**Observation:** for any $t \geq 1$, $X_t$ and $Y_t$ must be on the **same vertex**.

$$X_t = (v, i) \text{ and } Y_t = (v, j) \text{ (same vertex, different color)}$$

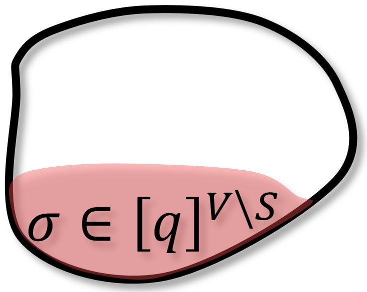
- Pick the same vertex $u \in V$ uniformly at random.
- Couple the colors on $u$ **optimally**, the coupling fails with probability

$$d_{TV}\big(\mu_u(\cdot \mid v \leftarrow i), \mu_u(\cdot \mid v \leftarrow j)\big) \leq \Psi(v, u). \quad \text{(Influence } v \to u\text{)}$$

**Spectral independence** → **$X, Y$ meet quickly in the coupling** → **local mixing** ⇒ [Alev and Lau 20] ⇒ **global mixing**

## Remark (conditional distributions)

- [AL20] requires local mixing on all **conditional distributions**.
- Our coupling also works for all **conditional distributions**.



# Spectral independence for coloring

**List coloring instance**

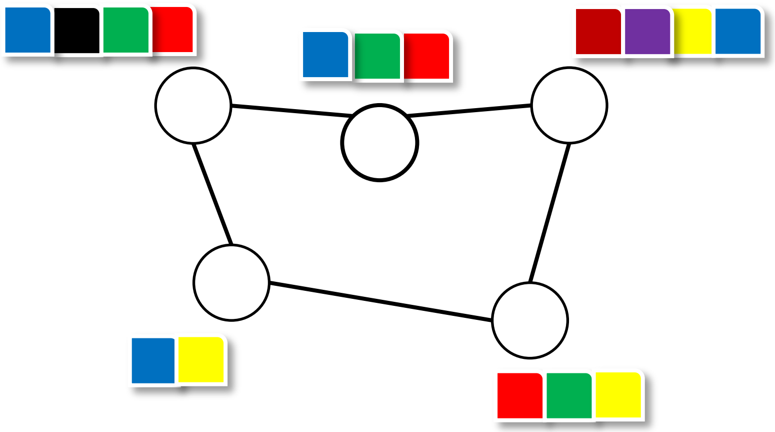
- graph $G = (V, E)$ with max degree $\Delta$;
- each vertex $v \in V$ has a color list $L_v$.

**Proper list coloring $\boldsymbol{X}$**

- $X_v \in L_v$ for all $v \in V$;
- $X_u \neq X_v$ for all $\{u, v\} \in E$.

**Gibbs distribution $\boldsymbol{\mu}$**

- uniform distribution over all proper list colorings.



**Theorem [this work]**

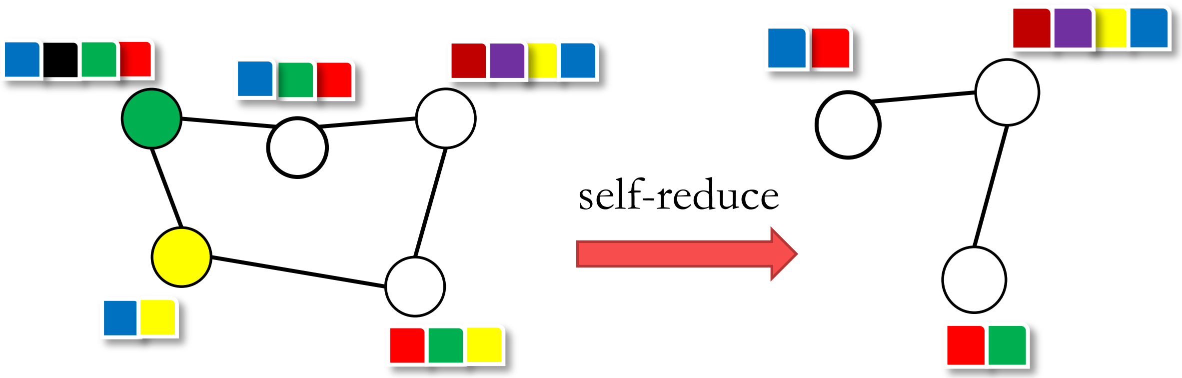
In **triangle-free graph**, if for all $v \in V$,

$$|L(v)| \geq (\alpha + \delta)\Delta \approx (1.763 + \delta)\Delta,$$

then under any pinning,

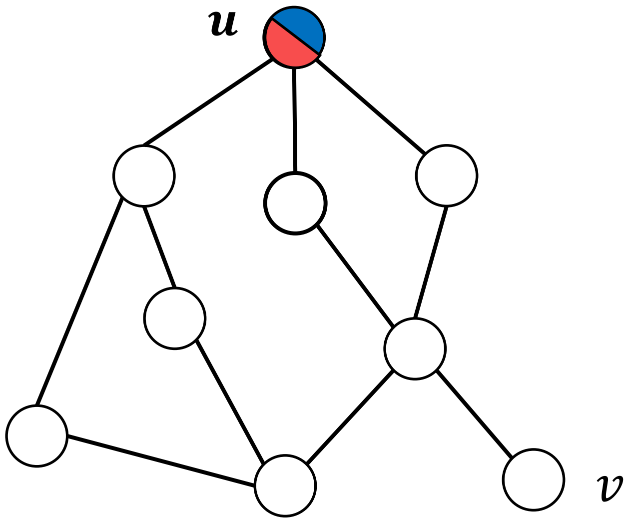
$$\textbf{one-to-all influence} = O\left(\frac{1}{\delta}\right),$$

→ $\mu$ is **spectrally independent** with $C = O\left(\frac{1}{\delta}\right)$.




closed under pinning

# Recursive coupling




**Influence from $u$ to $v$**

$$\mathrm{Inf}(u \to v) = \max_{i,j \in L(u)} d_{TV}\big(\mu_v(\cdot \mid u \leftarrow i), \mu_v(\cdot \mid u \leftarrow j)\big)$$

**One-to-all influence**

$$\sum_{v \in V \setminus \{u\}} \mathrm{Inf}(u \to v)$$

by **recursive coupling** **[Goldberg, Martin, Paterson 05]**

**Proof sketch**

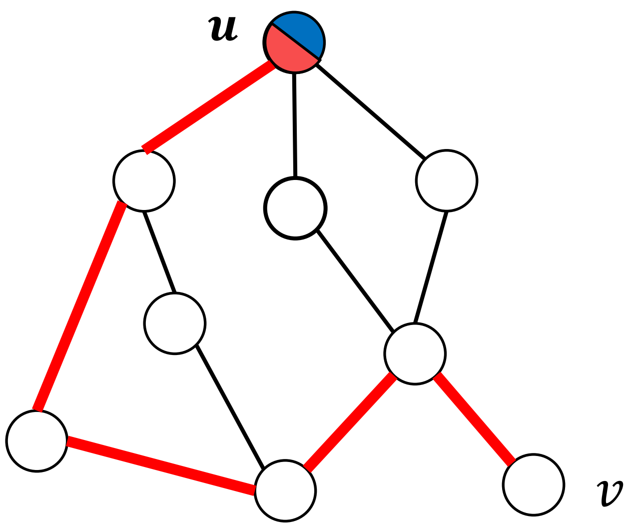
Construct a **coupling** $(c_v, c'_v)$ between $\mu_v(\cdot \mid u \leftarrow i)$ and $\mu_v(\cdot \mid u \leftarrow j)$

$$d_{TV}\big(\mu_v(\cdot \mid u \leftarrow i), \mu_v(\cdot \mid u \leftarrow j)\big) \le \Pr[c_v \ne c'_v]$$

Bound **one-to-all influence** by **coupling inequality**

$$\sum_{v \in V \setminus \{u\}} \mathrm{Inf}(u \to v) \le \sum_{v \in V \setminus \{u\}} \Pr[c_v \ne c'_v]$$

# Recursive coupling




- Staring from the "**disagreement vertex**" $u$.
- Coupling vertex one by one in a "**DFS-manner**".
- If the coupling on $v$ fails (i.e. $c_v \neq c'_v$)

  then **there is a path** $\mathcal{P}$ from $u$ to $v$

  **ALL vertices** in $\mathcal{P}$ **FAIL** in coupling.

Bound **one-to-all influence** by **enumerating all the paths from $u$**

$$\sum_{u \neq v} \mathrm{Inf}(u \to v) \leq \sum_{\text{all paths } P \text{ from } u} \text{Influence alone the path } P \leq O\left(\frac{1}{\delta}\right)$$

Triangle-free<br>Many colors → Coupling succeeds with high prob. → Bounded total influence

# Independent work

arXiv.org > cs > arXiv:2007.08058

**Computer Science > Data Structures and Algorithms**

*[Submitted on 16 Jul 2020]*

**Rapid Mixing for Colorings via Spectral Independence**

Zongchen Chen, Andreas Galanis, Daniel Štefankovič, Eric Vigoda

arXiv.org > cs > arXiv:2007.08091

**Computer Science > Data Structures and Algorithms**

*[Submitted on 16 Jul 2020]*

**Rapid mixing from spectral independence beyond the Boolean domain**

Weiming Feng, Heng Guo, Yitong Yin, Chihao Zhang

**Theorem** [Chen, Galanis, Štefankovič, Vigoda 20]

The Glauber dynamics for coloring is rapid mixing if $q \geq \alpha\Delta + 1$

- **different definition** of spectral independence
- **different method** to prove spectral independence for coloring

# Summary

- A definition a **spectral independence** for general distribution (generalize def. in [ALO20])
- **Rapid mixing** of **Glauber dynamics** from spectral independence
- Application: sampling uniform $q$-coloring on **triangle-free graph** when

$$q \geq (\alpha + \delta)\Delta \approx (1.763 + \delta)\Delta.$$

# Future work

- Improve the $n^{O(C)}$ mixing time for **general distribution**.
- Improve the $n^{O(1/\delta)}$ mixing time for **spin systems** (including coloring)

  $O(n \log n)$ optimal mixing for spin systems with

  - spectral independence and $\Delta = O(1)$ [CLV20, arXiv:2011.02075].
- **Better condition** for spectral independence
  - example: prove spectral independence for graph coloring with **fewer colors**.

Thank you