An FPRAS for two-terminal reliability in directed acyclic graphs

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All-terminal network reliability

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Input: an undirected graph $G = (V, E)$ and parameters $q_e \in (0,1)$

Output: the *probability* that G is ***connected*** if each edge $e \in E$ fails independently with prob. q_e

- #P-complete for exact computing [Valiant 79; Colbourn 87]
- FPRAS (fully poly-time approx. algorithm) exists [Guo and Jerrum18]

All-terminal network unreliability

Input: an undirected graph $G = (V, E)$ and parameters $q_e \in (0,1)$

Output: the *probability* that G is ***not connected*** if each edge $e \in E$ fails independently with prob. q_e

- #P-complete for exact computing [Valiant 79; Colbourn 87]
- FPRAS (fully poly-time approx. algorithm) exists [Karger 99; Cen, He, Li and Panigrahi 23]

s-t network reliability

Two-terminal network reliability

Input: a graph $G = (V, E)$ and parameters $q_e \in (0,1)$

a source node s and a sink node t

Output: the *probability* that $s \rightarrow_{G(p)} t$ if each edge $e \in E$ fails independently with prob. q_e

s can reach t in the remaining graph

$$\Pr[s \rightarrow_{G(p)} t] = \sum_{\substack{R \subseteq E: \\ s \to t \text{ in graph } (V,R)}} \prod_{e \in R} (1 - q_e) \prod_{e \notin R} q_e$$

Exact computing is #P-complete for ***directed*** / ***undirected*** / ***DAG*** / ***planar DAG*** graphs

Approximate s-t network reliability in DAGs

Two-terminal network reliability approximation

Input: a ***DAG (direct acyclic graph)*** $G = (V, E)$ and parameters $q_e \in (0,1)$

a source node s and a sink node t

an error bound $\varepsilon > 0$

Output: a ***random*** number $\hat{p}$ approximating s-t network reliability $p = \Pr\left[s \to_{G(p)} t\right]$

$$\Pr[\hat{p} \in (1 \pm \varepsilon)p] \geq \frac{2}{3}$$

Two-terminal network unreliability approximation

Output: a ***random*** number $\hat{q}$ approximating s-t network unreliability $1 - p$

$$\Pr[\hat{q} \in (1 \pm \varepsilon)(1 - p)] \geq \frac{2}{3}$$

Previous works

Direct Monte Carlo method (sample random subgraph and check reachability)

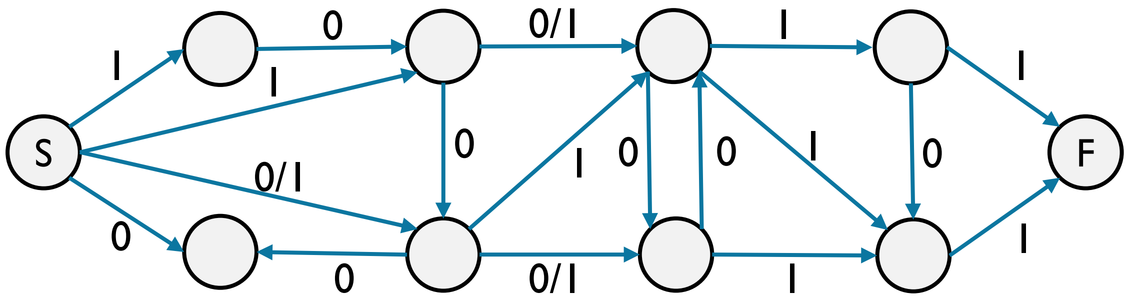
- the estimation is efficient if $p = \frac{1}{\mathrm{poly}(m)}$ (e.g. $q_e = O\left(\frac{\log m}{m}\right)$ is small, where $m = |E|$)

Improved analysis on lower bound of s-t reliability [Zenklusen and Laumanns 10]

- Improve the lower bound of reliability for some special DAGs

Approximate count accepting strings of **NFA** (nondeterministic finite automaton)

- Given n-state NFA, count #{distinct accepting strings} of length ℓ



Accepting String: 100011

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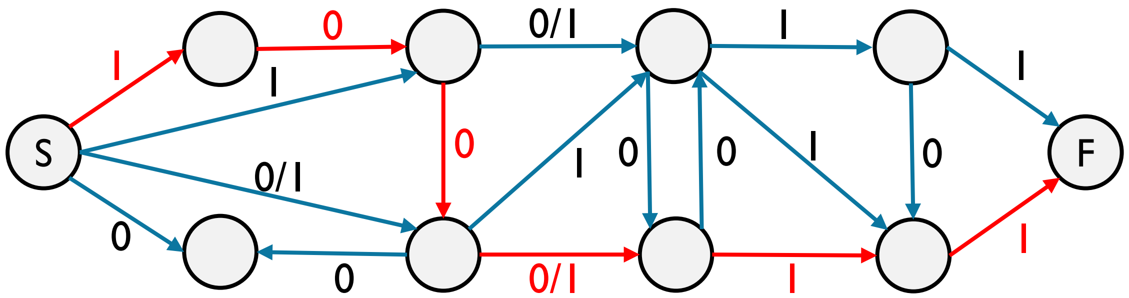

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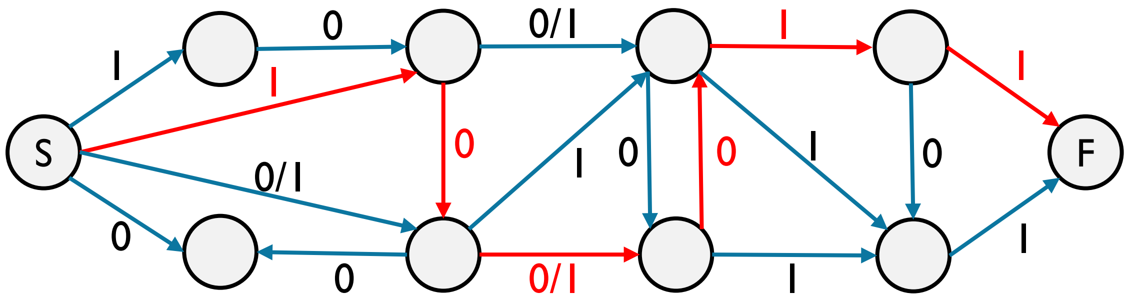
- FPRAS in time $\tilde{O}((n\ell)^{17})$ [Arenas, Croquevielle, Jayaram, Riveros, 21]



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- s-t reliability in DAGs can be reduced to #NFA [implied by Burtschick 95] [explicitly in Amarilli, Bremen and Meel 24]

  ➡ An FPRAS for s-t reliability in DAG (running time is a huge polynomial)

Very recently, improved FPRAS for #NFA [Meel, Chakraborty and Mathur 24]

- An FPRAS for s-t reliability in time $\tilde{O}(m^{19})$ for $q_e = \frac{1}{2}$ via **standard black-box reduction**

Our results

There is an ***FPRAS*** for two-terminal network ***reliability*** in DAGs in time

$$\tilde{O}(n^6 m^4 \max\{m^4, \varepsilon^{-4}\}) \text{ , where } n = |V| \text{ and } m = |E|$$

- technique inspired by [Arenas, Croquevielle, Jayaram and Riveros 21]
- running time can be further reduced by combining our technique with very recent technique for #NFA [Meel, Chakraborty and Mathur 24]

There is ***no FPRAS*** for two-terminal network ***unreliability*** in DAGs unless there is an FPRAS for ***#BIS*** problem

#BIS: counting the number of independent sets in bipartite graphs conjectured to has no FPRAS

Some basic settings for this talk

Two-terminal network reliability approximation

Input: a ***DAG (direct acyclic graph)*** $G = (V, E)$ and parameters $q_e \in (0,1)$

a source node s and a sink node t

an error bound $\varepsilon > 0$

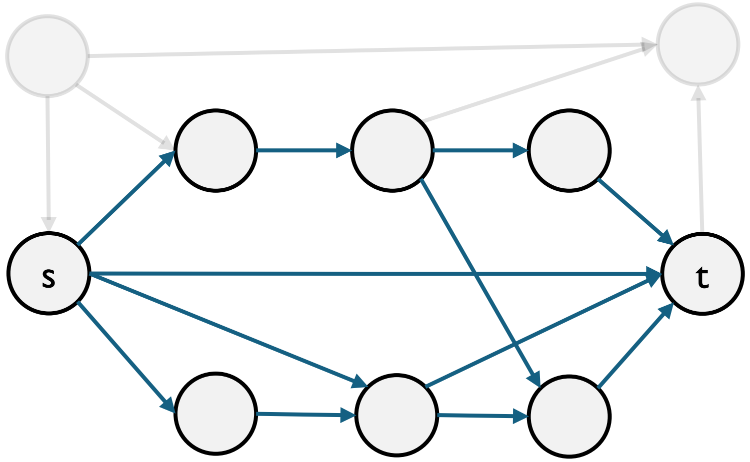
Output: a ***random*** number $\hat{p}$ approximating s-t network reliability $p = \Pr[s \to_{G(p)} t]$

1) For any vertex v in DAG $G = (V, E)$ assume that
   v can be reached from s and v can reach t

2) For any edge $e \in E, q_e = \frac{1}{2}$
   general $q_e \in (0,1)$ can be solved with ***very small tweaks***

$$p = \frac{\#\{\text{subgraphs such that } s \text{ can reach } t\}}{2^m}$$

counting problem

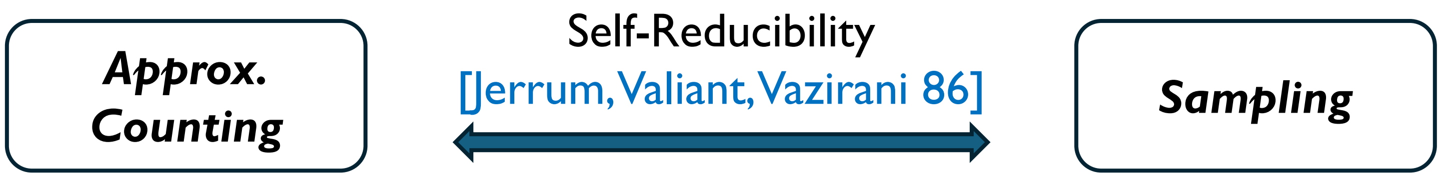


Sampling and counting

Two-terminal network reliability in DAGs

$$\Omega = \{\text{subgraphs of } G \text{ such that } s \text{ can reach } t\}$$

- ***Counting Problem (network reliability)***: estimate the size $\#\Omega$
- ***Sampling Problem***: draw random subgraphs from Ω uniformly at random



Sampling algorithm: MCMC (Metropolis chain)

Maintain a subset of edges $R \in \Omega$ (s can reach t in subgraph (V, R))

- Pick an edge $e \in E$ uniform at random
- If $e \in R$ and $R - e \in \Omega$, let $R \leftarrow R - e$
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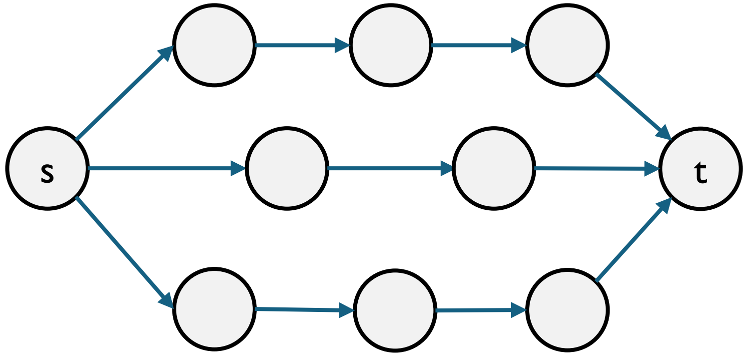

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Approx. Counting $\longleftrightarrow$ ***Sampling***

Self-Reducibility
[Jerrum, Valiant, Vazirani 86]

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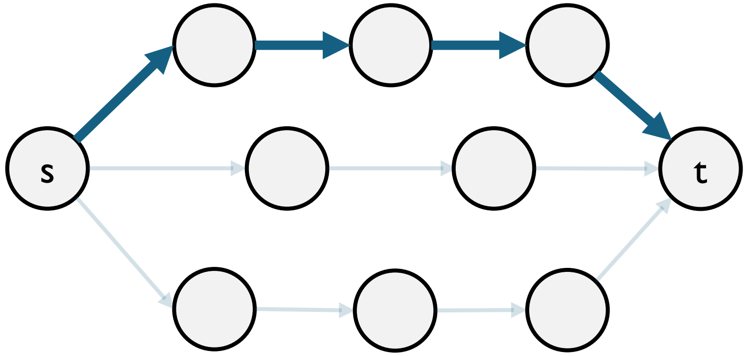

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Approx. Counting ⟷ ***Sampling***

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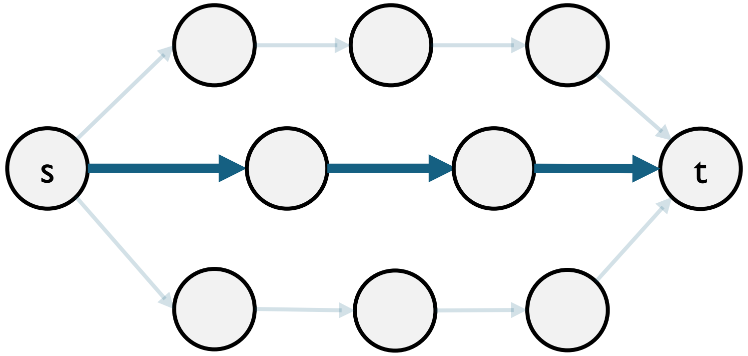
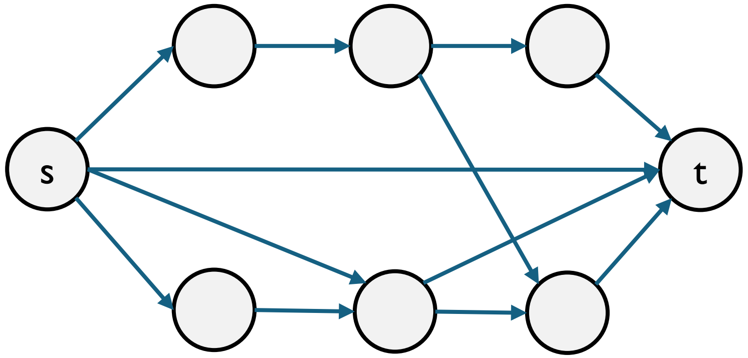


Our Algorithm



sort all vertices according to topological order $s = v_n \succ v_{n-1} \succ \cdots \succ v_1 = t$

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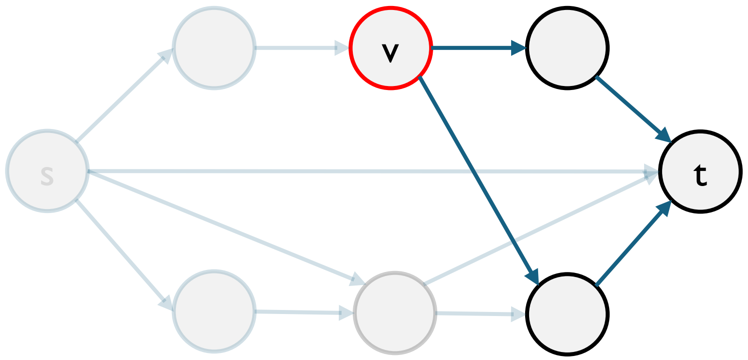

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- G_v: subgraph containing all vertices that can be reached from $v \in V$

$$\Omega_v = \{\text{subgraphs of } G_v \text{ such that } v \text{ can reach } t\}$$

- Z_v: a **counting estimate** of the size $\#\Omega_v$
- S_v: a set of **samples**, where each $H \in S_v$ is ***uniform random sample*** from Ω_v

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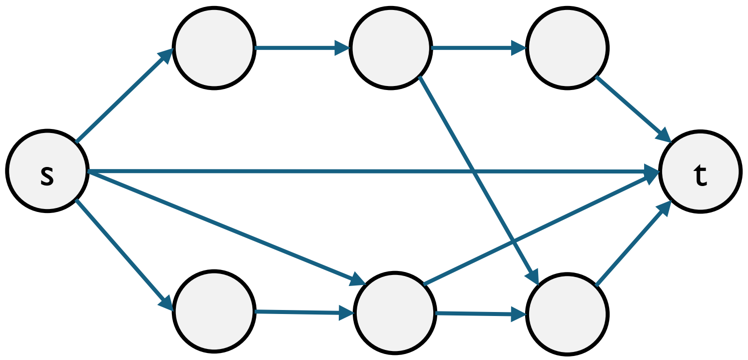

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Compute (Z_v, S_v) for v from v_1 to v_n via ***dynamic programming*** + ***Monte Carlo***

Framework appeared in [Gore, Jerrum, Kannan, Sweedyk, and Mahaney 97] [Arenas, Croquevielle, Jayaram and Riveros 21]

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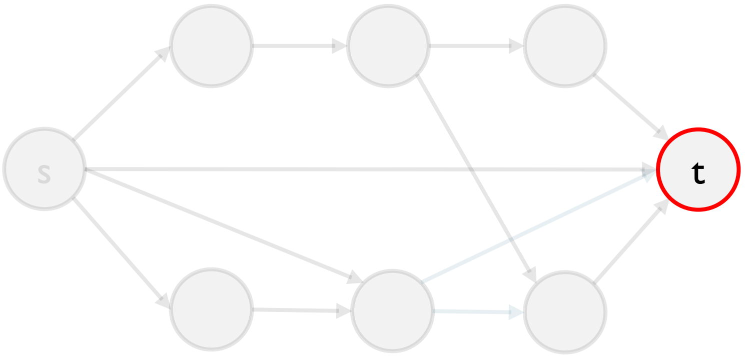

base case $v = t$ is trivial

sort all vertices according to topological order $s = v_n \succ v_{n-1} \succ \cdots \succ v_1 = t$

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Estimate the count via Karp-Luby

- A vertex $v \in V$ and a set of ***neighbors*** $v_1, v_2, v_3, \dots, v_d$

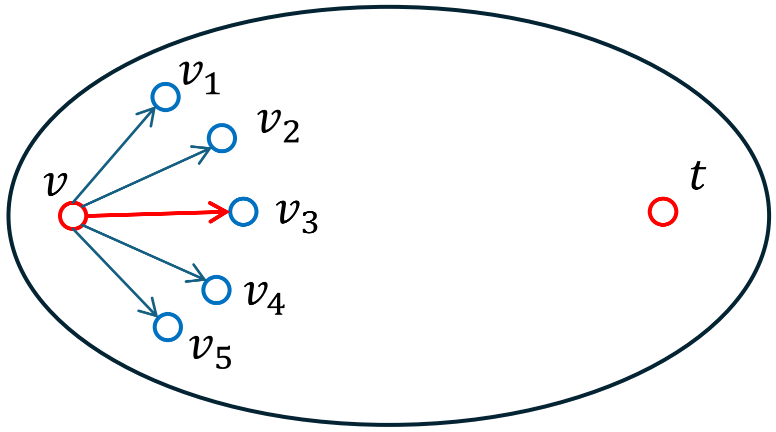
$$\Omega_v = \{\text{subgraphs of } G_v \text{ s.t. } v \to t\}$$

- For any neighbor v_i,

$$\Omega_i = \{\text{subgaphs of } G_v \text{ s.t. } v \to v_i \to t\}$$

- Vertex v must reach t through some neighbors

$$\Omega_v = \cup_{i=1}^{d} \Omega_i$$



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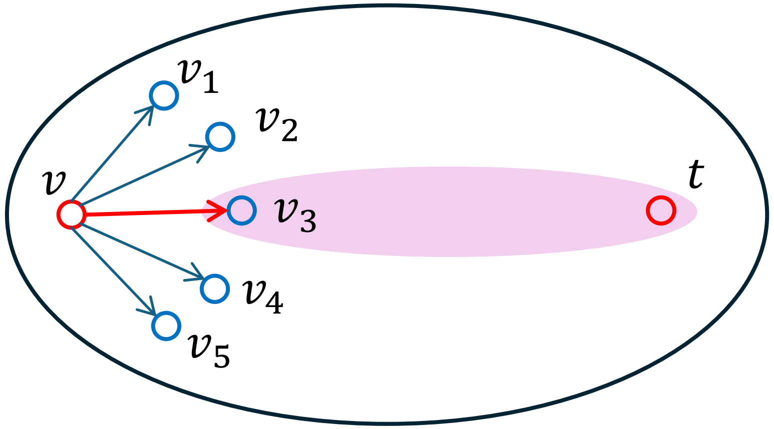

Karp-Luby's method for estimating the size of union [Karp and Luby 83]

- Sample an index $i \in [d]$ with prob $\propto |\Omega_i|$ ➡ estimate $|\Omega_i|$ using $Z_{v_i} \approx |\Omega_{v_i}|$
- Sample a subgraph H from Ω_i uniformly at random ➡ take a sample $H' \in S_{v_i}$ and modify $H' \to H$
- $X \in \{0,1\}$ indicate whether i is the first set containing H ➡ do DFS search in H

$$\mathbb{E}[X] = \frac{|\Omega_v|}{\sum_{i=1}^{d} |\Omega_i|} \geq \frac{1}{d} \qquad |\Omega_v| = \left(\sum_{i=1}^{d} |\Omega_i|\right) \mathbb{E}[X]$$

generate ind. X and take average

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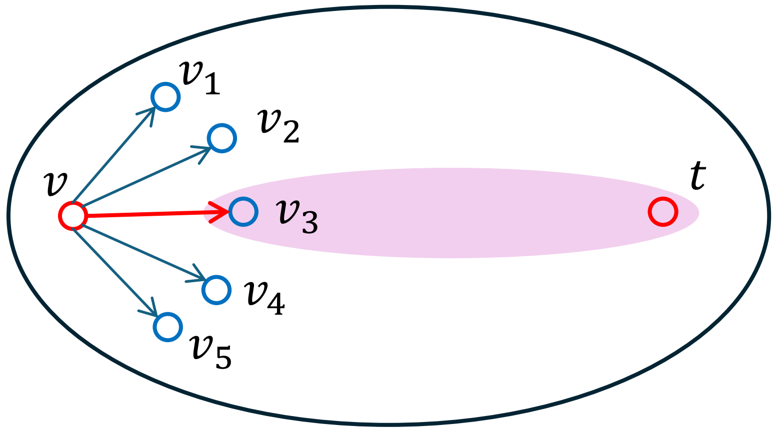

- Apply ***Karp-Luby*** to compute an estimate Z_v such that

$$\Pr\left[Z_v \in \left(1 \pm \mathrm{poly}\left(\frac{\varepsilon}{m}\right)\right)|\Omega_v|\right] \geq \frac{2}{3}$$

- Apply ***median-trick*** to boost the successful prob.

$$2/3 \to 1 - \exp(-\mathrm{poly}(m))$$

- ***Higher accuracy*** and ***higher successful probability*** require ***more samples*** in set S_{v_i}

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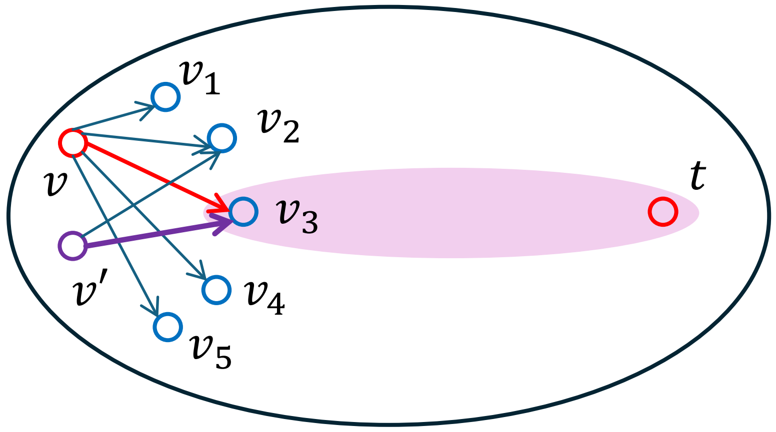

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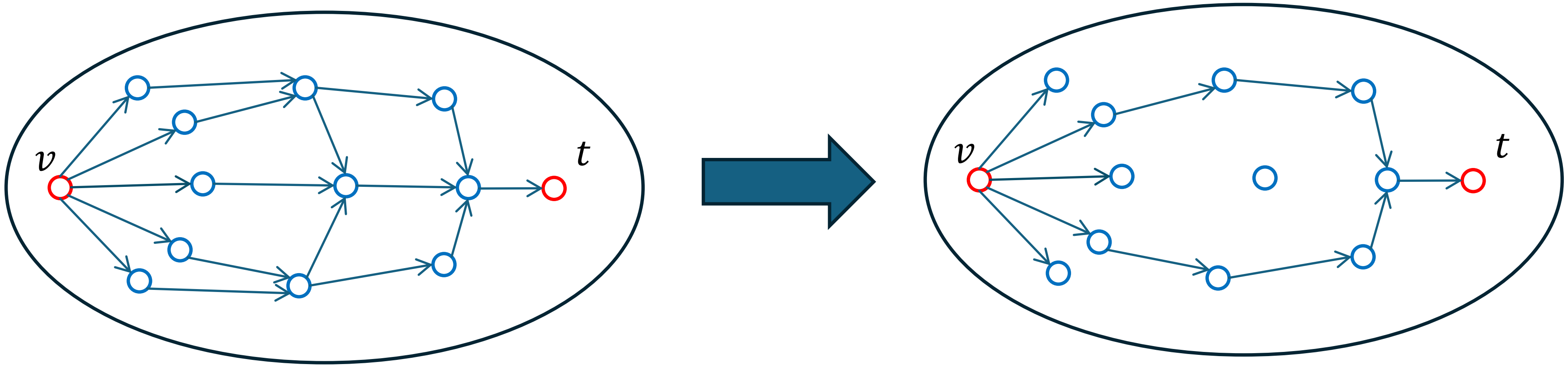
- Different vertices v, v' may have the same neighbor v_i, we ***reuse*** S_{v_i} when computing Z_v and $Z_{v'}$

Generate the samples via self-reduction



graph G_v random subgraph of G_v s.t. $v \to t$

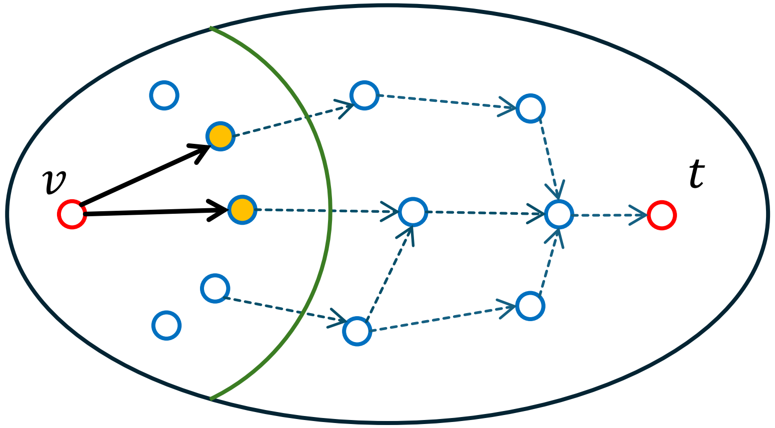
- Go through all edges $e_1, e_2, \ldots, e_m$ in graph G_v in some ordering
- For each edge e_i, sample $X(e_i) \in \{0,1\}$ that indicates whether e_i in random subgraph
- Compute the ***conditional marginal probability***

$$p(e_i) = \Pr[\, X(e_i) = 1 \mid X(e_1), X(e_2), \ldots, X(e_{i-1}) \,]$$

- Set $X(e_i) = 1$ with prob. $p(e_i)$ and set $X(e_i) = 0$ with prob. $1 - p(e_i)$
- Repeat the processing for $\mathrm{poly}(m/\epsilon)$ times to generate $\mathrm{poly}(m/\epsilon)$ samples

Generate the samples via self-reduction

- Suppose we have sampled $X(e_i)$ for $i \le \ell$
- Let $\mathcal{E}$ be set of edges e_i s.t. $X(e_i) = 1$
- $\Lambda = \{w \in V: v \to w \text{ through edges in } \mathcal{E}\}$



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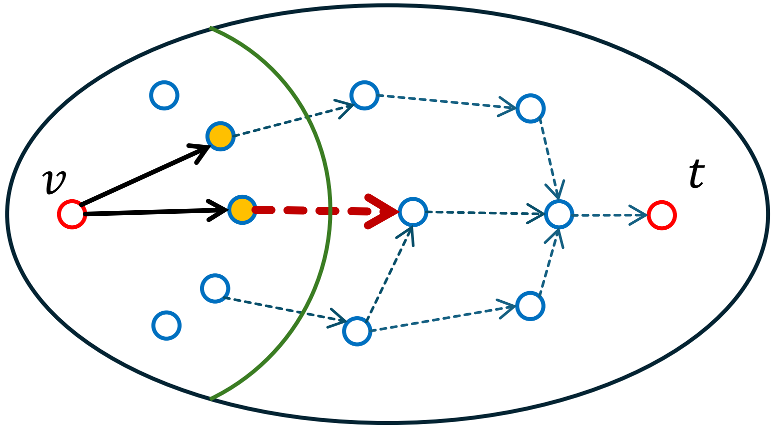
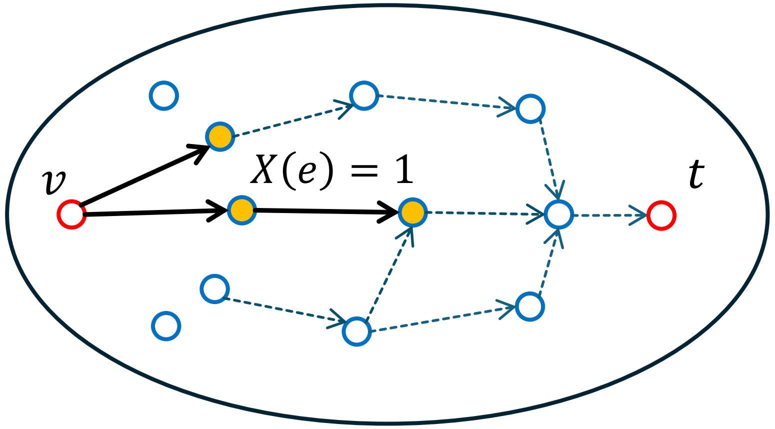


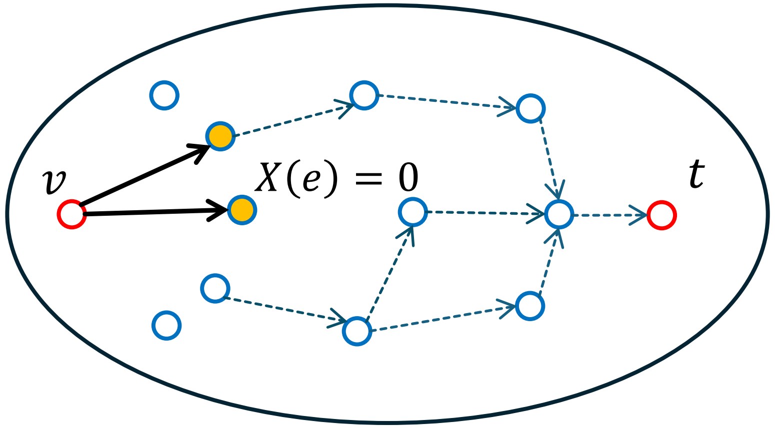
- Pick the edge $e = (u, w)$ from Λ to the ***out-boundary*** $\partial\Lambda$ where w has the largest topological order

$$\partial\Lambda = \{w \notin \Lambda: \exists u \in \Lambda\ s.t.\ (u, w) \in E\}$$



Prob ∝ # {subgraphs v can reach t}

OR



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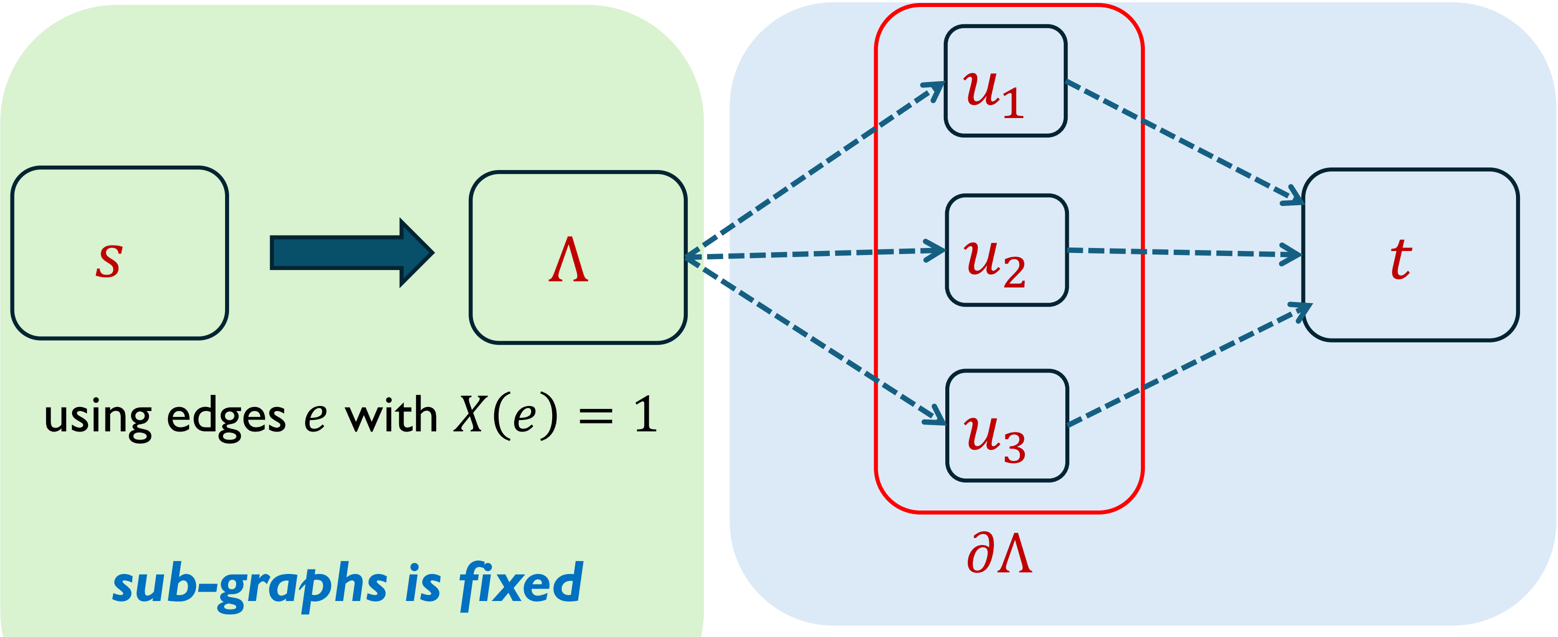

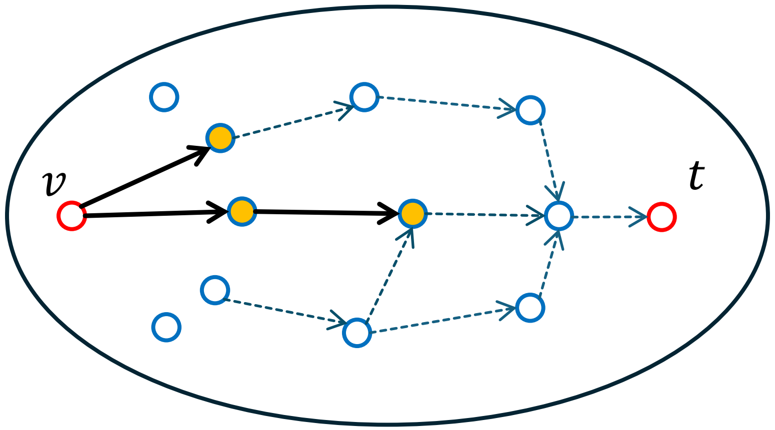

Approx, count # {subgraphs s.t. Λ can reach t through $\partial\Lambda$}

$$\Omega = \bigcup_{u \in \partial\Lambda} \Omega_u$$

- $\Omega =$ {subgraphs s.t. Λ can reach t} is the set we want to count
- $\Omega_u =$ {subgraphs s.t. Λ can reach t through ***u***}
- Run ***Karp-Luby algorithm*** by ***reusing*** the samples in nodes at $\partial\Lambda$

Sampling Algorithm at Each Node

approximate counting via Karp-Luby (reusing samples) → estimate ***conditional marginal probability*** → Generate samples by sampling edges one-by-one

↑ also used for compute the ***count estimate Z_v*** at each node

- Sort all vertices in topological ordering $t = v_1 \prec v_2 \prec \cdots \prec v_n = s$
- G_v: subgraph containing all vertices that can be reached from $v \in V$

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High-level analysis of the algorithm

Challenge: ***reusing*** samples introduces complicated ***correlations***

Key Property: the algorithm only use samples for approx. counting (estimate marginal prob.)

approximate counting via Karp-Luby (reusing samples)	→	estimate *conditional marginal prob. p*	→	Generate samples by sampling edges one-by-one

use ***fresh randomness*** to sample $X(e)$ with $\Pr[X(e) = 1] = p$

- The algorithm is correct if random samples ***correctly*** estimate counts or marginal prob.

$\text{poly}(m)$ number samples	**Chernoff + median trick** →	estimation is correct with prob. $1 - \exp(-m)$

- Conditional on estimate is correctly only bias random samples with ***exp-small*** error
- Show the whole algorithm is correct by an **induction proof**

Analysis of the algorithm

When algorithm scans u, the following ***good event*** happens with ***high probability***

A. Random samples on all vertices $v \preccurlyeq u$ are ***accurate***

$$TV(\text{random samples in } S_v, \text{unifrom distribution of } \Omega_v) \leq \exp(-\Omega(m))$$

B. If we run Karp-Luby on any $\partial\Lambda \subseteq \{v \mid v \preccurlyeq u\}$, the answer is ***accurate***

$$\text{Estimate } \cup_{w\in\partial\Lambda} \Omega_w \text{ with } \left(1 \pm \text{poly}\left(\frac{\epsilon}{m}\right)\right) \text{ error}$$

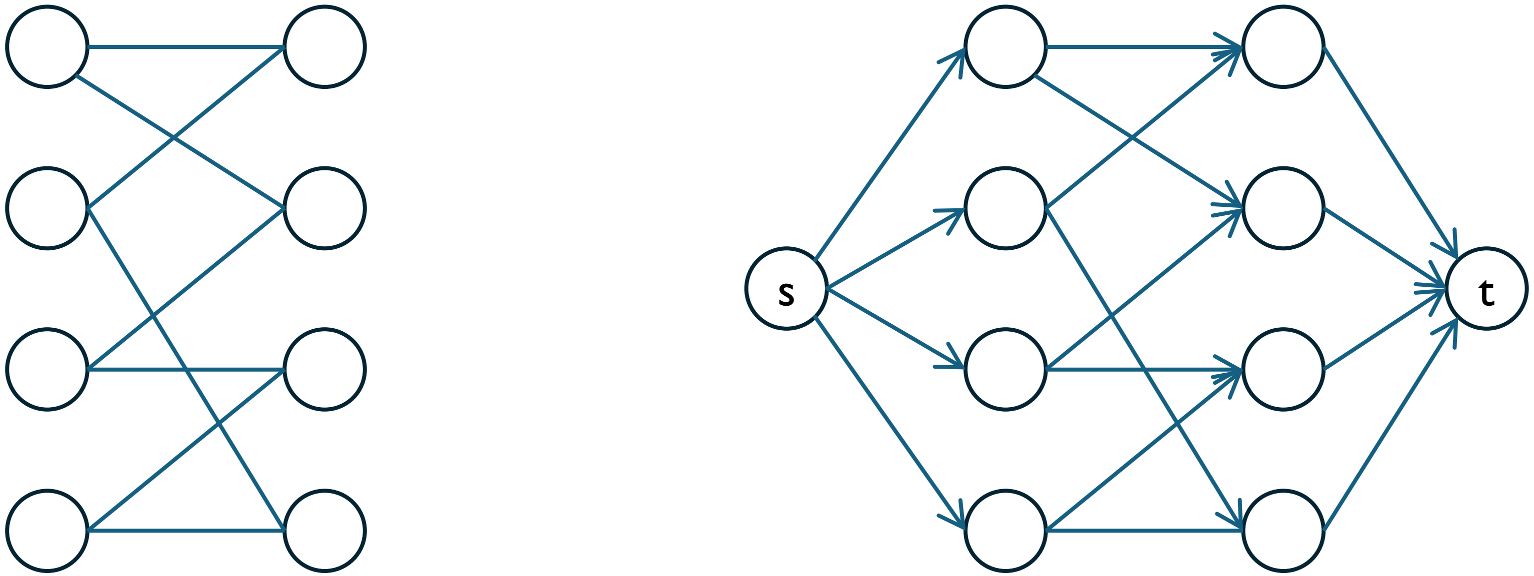
the error becomes worse during induction
the final error can be controlled

The induction proof

- If (B) holds for all $v \prec u$, then the algorithm at u generates ***prefect samples*** (via a filter)
- If (A)(B) holds for all $v \prec u$, then (B) holds for u with prob. $1 - O(\mathrm{e}^{-\Omega(m)})$
- Conditional on (B) holds for u only bias the sample with $O(e^{-\Omega(m)})$ TV-distance error

Proof of hardness

There is ***no FPRAS*** for two-terminal network ***unreliability*** in DAGs
unless there is an FPRAS for ***#BIS*** problem



#BIS $\leq$ vertex unreliability

Proof of hardness

There is ***no FPRAS*** for two-terminal network ***unreliability*** in DAGs
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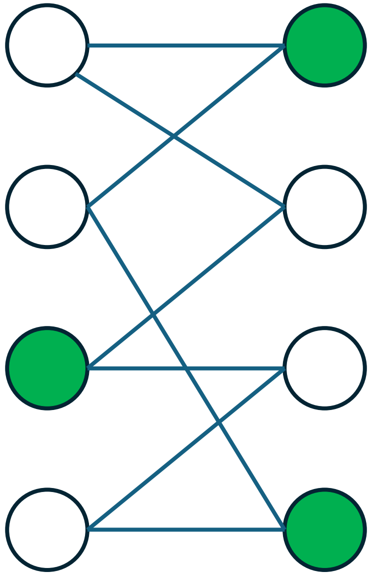

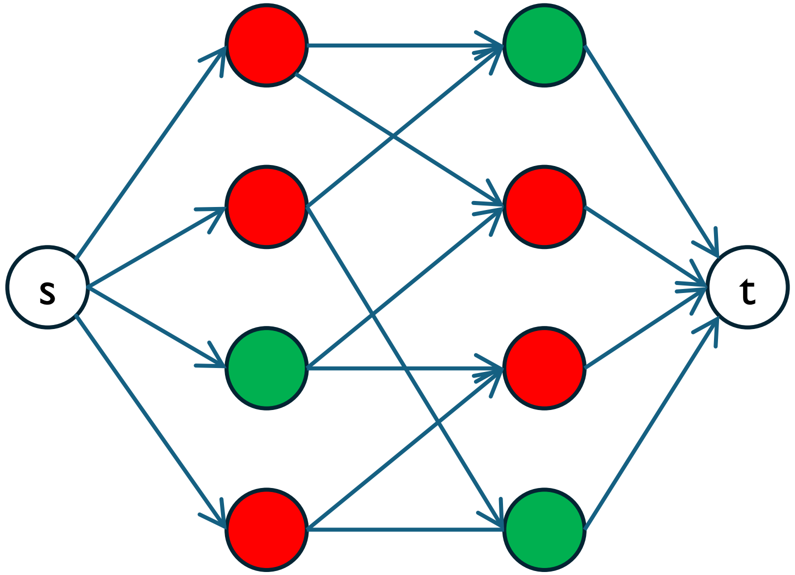

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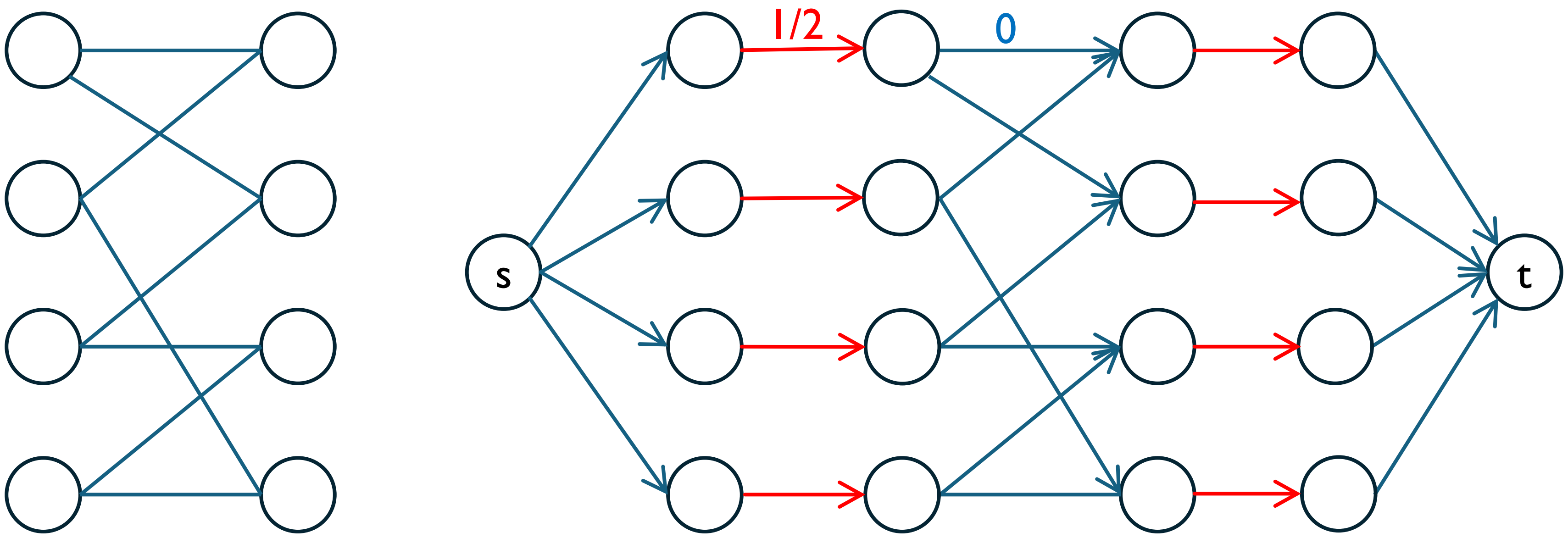

#BIS $\leq$ edge unreliability

Open Problems

- ***Faster*** algorithm for s-t reliability in DAGs
- ***Simple*** algorithm for s-t reliability in DAGs
- FPTAS (***deterministic***) algorithm for s-t reliability in DAGs
- Algorithm or hardness for approximating s-t reliability in ***undirected / directed*** graphs
- Simple or faster FPRAS/FPTAS for ***#NFA***

Thank you
Q&A