# Dynamic Sampling from Graphical Models

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## Abstract

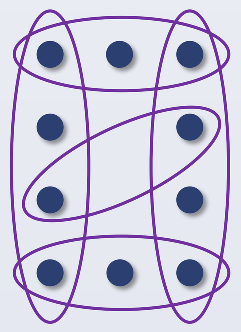
We study the problem of sampling from a graphical model when the model itself is changing dynamically with time.

- We give an algorithm that can sample dynamically from a broad class of graphical models efficiently.
- We give an equilibrium condition that guarantees the correctness of the dynamic sampling.

## Graphical Model

Graphical models arise in a variety of disciplines ranging from statistical physics, machine learning, statistics, to theoretical computer science. A graphical model instance is specified by a tuple $\mathcal{I} = (V, E, Q, \Phi)$:

- variable set (vertex set) $V$;
- constraint set (edge set) $E \subseteq 2^V$;
- finite domain $Q$;
- factors (weight functions)
  $\Phi = (\phi_v)_{v \in V} \cup (\phi_e)_{e \in E}$
  - each $\phi_v: Q \to \mathbb{R}_{\geq 0}$;
  - each $\phi_e: Q^e \to \mathbb{R}_{\geq 0}$;



hyper graph $H = (V, E)$

- Gibbs distribution $\mu$ over $Q^V$:

$$\forall \sigma \in Q^V, \quad \mu(\sigma) \propto \prod_{v \in V} \phi_v(\sigma_v) \prod_{e \in E} \phi_e(\sigma_e).$$

### Example: Ising model $\mathcal{I} = (V, E, \boldsymbol{\beta})$

- graph $G = (V, E)$;
- finite domain $Q = \{-1, +1\}$;
- inverse temperature $\boldsymbol{\beta} = (\beta_e)_{e \in E}$, each $\beta_e \in \mathbb{R}_{\geq 0}$;
- Gibbs distribution $\mu$ over $\{-1, +1\}^V$:

$$\forall\, \sigma \in \{-1, +1\}^V, \quad \mu(\sigma) \propto \prod_{e = (u,v) \in E} \exp(\beta_e \sigma_u \sigma_v);$$

- uniqueness condition

$$\forall e \in E: \quad \exp(-2|\beta_e|) > 1 - \frac{2}{\Delta}.$$

### Example: hardcore model $\mathcal{I} = (V, E, \boldsymbol{\lambda})$.

- graph $G = (V, E)$;
- finite domain $Q = \{0, 1\}$;
- fugacity $\boldsymbol{\lambda} = (\lambda_v)_{v \in V}$, each $\lambda_v \in \mathbb{R}_{\geq 0}$;
- Gibbs distribution $\mu$ over $\{0,1\}^V$: $\forall\, \sigma \in \{0,1\}^V$,

$$\mu(\sigma) \propto \begin{cases} \prod_{v \in I(\sigma)} \lambda_v & \text{if } I(\sigma) \text{ is an independent set,} \\ 0 & \text{if } I(\sigma) \text{ is not an independent set,} \end{cases}$$

where $I(\sigma) = \{ v \in V \mid \sigma_v = 1 \}$;

- uniqueness condition

$$\forall v \in V: \quad \lambda_v < \frac{(\Delta - 1)^{\Delta - 1}}{(\Delta - 2)^{\Delta}} \approx \frac{e}{\Delta - 2}.$$

## Dynamic Sampling Problem

Given: dynamic graphical model and current sample.

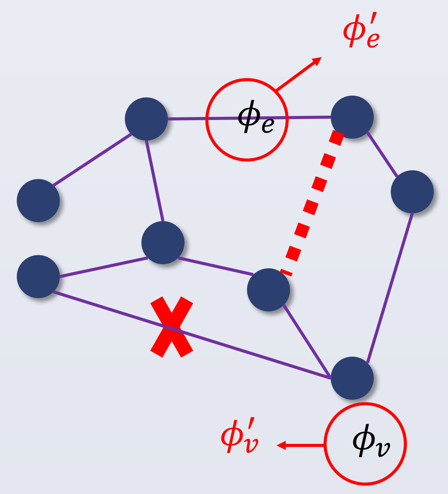
Main question : **"Can we obtain a sample from an updated graphical model with a small incremental cost?"**

### Updates of graphical model

- add/delete constraints;
- change factors $\phi_v \to \phi'_v$, $\phi_e \to \phi'_e$;
- add/delete independent variables.

An update of graphical model $\mathcal{I} = (V, E, Q, \Phi)$ is represented by a pair $(D, \Phi_D)$:

- $D \subseteq V \cup 2^V$: updated variables and constraints;
- $\Phi_D := (\phi_v)_{v \in D \cap V} \cup (\phi_e)_{e \in D \cap 2^V}$: new factors.




Input graphical model $\mathcal{I} = (V, E, Q, \Phi)$ → **Update** $(D, \Phi_D)$ → updated graphical model $\mathcal{I}' = (V, E', Q, \Phi')$

### Dynamic sampling from graphical model

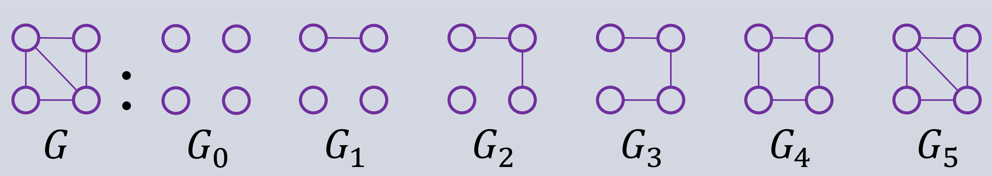
- **Input**: a graphical model $\mathcal{I}$, a sample $\boldsymbol{X} \sim \mu_{\mathcal{I}}$ and an update $(D, \Phi_D)$ that modifies $\mathcal{I}$ to $\mathcal{I}'$.
- **Output**: a sample $\boldsymbol{X}' \sim \mu_{\mathcal{I}'}$.

**Offline adversary**: the update $(D, \Phi_D)$ is independent with the input random sample $\boldsymbol{X} \sim \mu_{\mathcal{I}}$.

### Motivation

**Approximate counting** [Jerrum, Valiant, Vazirani, 1986]

- Given a graph $G = (V, E)$, count #{independent sets of $G$}.
- **Self reduction**: a sequence of graphs $G_0, G_1, \ldots, G_{|E|}$:




- Counting → Sampling uniform independent sets.

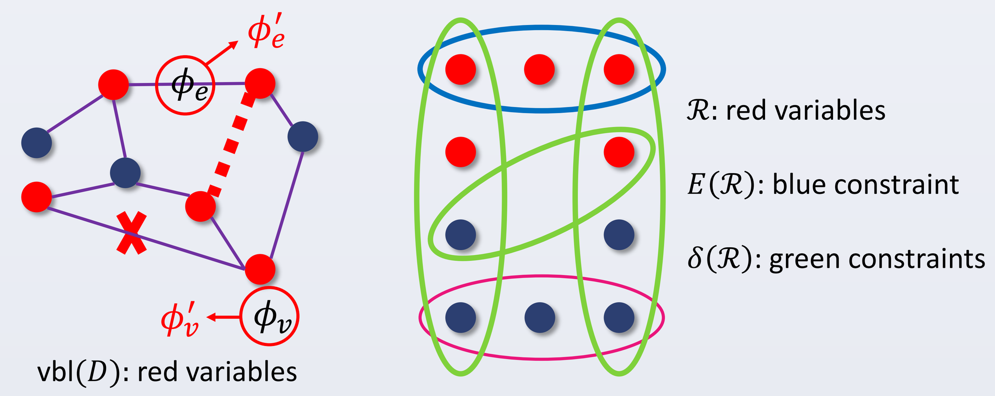
**Inference/learning tasks**

- online learning with dynamic or streaming data;
- dynamic graphical models e.g. videos.

## Dynamic Sampler

### Notations

- Update of graphical model $(D, \Phi_D)$.
- $\text{vbl}(D) := (D \cap V) \cup (\bigcup_{e \in D \cap 2^V} e)$: variables **involved** by the update $(D, \phi_D)$ :
  - updated variables;
  - variables incident to updated constraints.




- Subset of variables $\mathcal{R} \subseteq V$:
  - **internal** constraints $E(\mathcal{R}) := \{e \in E \mid e \subseteq \mathcal{R}\}$
  - **boundary** constraints $\delta(\mathcal{R}) := \{ e \in E \setminus E(\mathcal{R}) \mid e \cap \mathcal{R} \neq \emptyset \}$
  - **incident** constraints $E^+(\mathcal{R}) := E(\mathcal{R}) \cup \delta(\mathcal{R})$.

### The Algorithm

**Assumption: normalized factors** $\Phi = (\phi_v)_{v \in V} \cup (\phi_e)_{e \in E}$
each $\phi_v: Q \to [0,1]$ is a distribution over $Q$;
each $\phi_e: Q^e \to [0,1]$.

**Dynamic Sampler**
**Input**: a graphical model $\mathcal{I}$ and a sample $\boldsymbol{X} \sim \mu_{\mathcal{I}}$;
**Update**: an update $(D, \phi_D)$ that modifies $\mathcal{I} \to \mathcal{I}'$;

- apply changes $(D, \phi_D)$ to current graphical model $\mathcal{I}$;
- $\mathcal{R} \leftarrow \text{vbl}(D)$;
- **While**$(\mathcal{R} \neq \emptyset)$
  - $(\boldsymbol{X}, \mathcal{R}) \leftarrow$**Local-Resample**$(\boldsymbol{X}, \mathcal{R})$;
- **Return** $\boldsymbol{X}$;

**Local-Resample**$(\boldsymbol{X}, \mathcal{R})$:

- each $e \in E^+(\mathcal{R})$ computes $\kappa_e$; ⟵ first , compute $\kappa_e$
- each $v \in \mathcal{R}$ resamples $X_v \sim \phi_v$; ⟵ then , update $X_{\mathcal{R}}$
- each $e \in E^+(\mathcal{R})$ samples $F_e \in \{0,1\}$ independently s.t.
  $\Pr[F_e = 0] = \kappa_e \phi_e(X_e)$; ⟵ depend on both old and new samples
- $\boldsymbol{X}' \leftarrow \boldsymbol{X}$ and $\mathcal{R}' \leftarrow \bigcup_{e \in E^+(\mathcal{R}): F_e = 1} e$;
- **Return** $(\boldsymbol{X}', \mathcal{R}')$;

$$\kappa_e := \frac{1}{\phi_e(X_e)} \min_{\substack{y \in Q^e \\ y_{e \cap \mathcal{R}} = X_{e \cap \mathcal{R}}}} \phi_e(y)$$

(with the convention $\frac{0}{0} = 1$).

$\kappa_e$: the minimum value of $\phi_e(y)$ conditioning on the assignment of $y$ on $e \cap \mathcal{R}$ is fixed as $X_{e \cap \mathcal{R}}$.

**Properties:**

- for each $e \in E(\mathcal{R})$, $\kappa_e = 1$;
- for each $e \in \delta(\mathcal{R})$, $\kappa_e \leq 1$.

## Our Results

**Theorem: Correctness**

The dynamic sampler outputs the correct sample $\boldsymbol{X} \sim \mu_{\mathcal{I}'}$ .

*guaranteed by the* ***equilibrium condition***

**Features of the Algorithm**

dynamic, exact sampling, Las Vegas, distributed/parallel.

**Theorem: Fast Convergence**

- $d := \max_{e \in E} |\{e' \in E \setminus \{e\} \mid e \cap e' \neq \emptyset\}|$: the maximum degree of the dependency graph
- $\forall e \in E: \quad \min \phi_e \geq \sqrt{1 - \frac{1}{d+1}}$,

$\Longrightarrow$ the cost of the dynamic sampler:

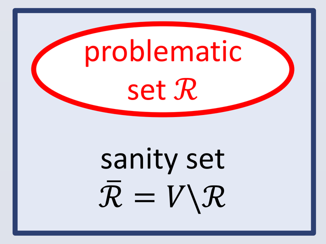
- $O(\log|D|)$ iterations in expectation;
- $O(|D|)$ resamplings in expectation.

Better results on concrete graphical models:

- Ising model: $\forall e \in E: \quad \exp(-2|\beta_e|) \geq 1 - \frac{1}{2.221\Delta + 1}$;
- Hardcore model: $\forall v \in V: \quad \lambda_v \leq \frac{1}{\sqrt{2}\Delta - 1}$.

## Equilibrium Condition

The dynamic sampler maintains a random pair $(\boldsymbol{X}, \mathcal{R}) \in Q^V \times 2^V$.




$\mathcal{R}$: current **resample set** that contains the problematic variables to be resampled;
$\bar{\mathcal{R}}$: current **sanity set** that contains the non-problematic variables.

**Conditional Gibbs property:**
A random pair $(\boldsymbol{X}, \mathcal{R}) \in Q^V \times 2^V$ is conditionally Gibbs w.r.t. $\mu$ if conditioning on any $\mathcal{R} \subseteq V$ and any assignment $\sigma \in Q^{\mathcal{R}}$ of $X_{\mathcal{R}}$, the distribution of $X_{V \setminus \mathcal{R}}$ is precisely $\mu^{\sigma}_{V \setminus \mathcal{R}}$.

$\mu^{\sigma}_{V \setminus \mathcal{R}}$: marginal distribution of $\mu$ on $V \setminus \mathcal{R}$ conditioning on $\sigma$.

When $\mathcal{R} = \emptyset$, the random sample $\boldsymbol{X} \sim \mu$.

**Resampling chain**
The resampling algorithm is a Markov chain over $Q^V \times 2^V$ with transition matrix $P: (\boldsymbol{X}, \mathcal{R}) \to (\boldsymbol{X}', \mathcal{R}')$.

**Equilibrium condition for resampling chain:**
If $(\boldsymbol{X}, \mathcal{R})$ is conditionally Gibbs w.r.t. $\mu$, then $(\boldsymbol{X}', \mathcal{R}')$ is also conditionally Gibbs w.r.t. $\mu$.

The condition is established by verifying **equation system**:
$\forall\ S, T \subseteq V, \sigma \in Q^{V \setminus S}$ and $\tau \in Q^{V \setminus T}$,

$$\forall y \in Q^V, y_{V \setminus T} = \tau: \sum_{\substack{x \in Q^V \\ x_{V \setminus S} = \sigma}} \mu^{\sigma}_S(x_S) \cdot P\big((x, S), (y, T)\big) = C(S, \sigma, T, \tau) \cdot \mu^{\tau}_T(y_T).$$

Dynamic Sampling Algorithm — a solution to ⟶ Equation System Equilibrium Condition