Rapid mixing of Glauber dynamics via spectral independence for all degrees

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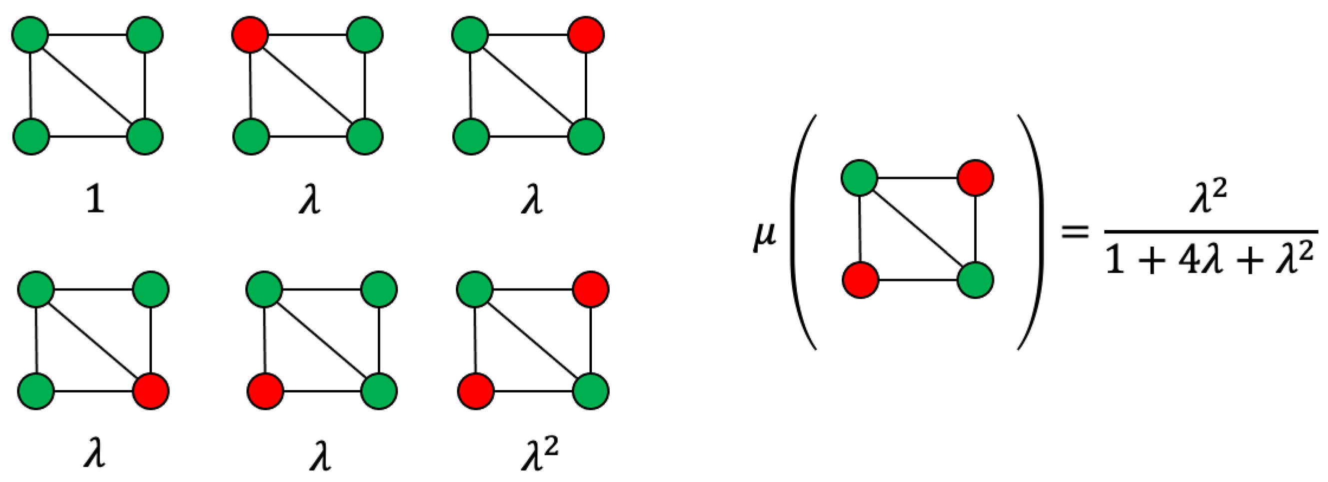
Hardcore model

Parameters: graph $G = (V, E)$ and fugacity $\lambda > 0$.
State space: $\Omega \subseteq \{-,+\}^V$ s.t. vertices with $+$ form an independent set.
Gibbs distribution: μ over all independent sets Ω in G:

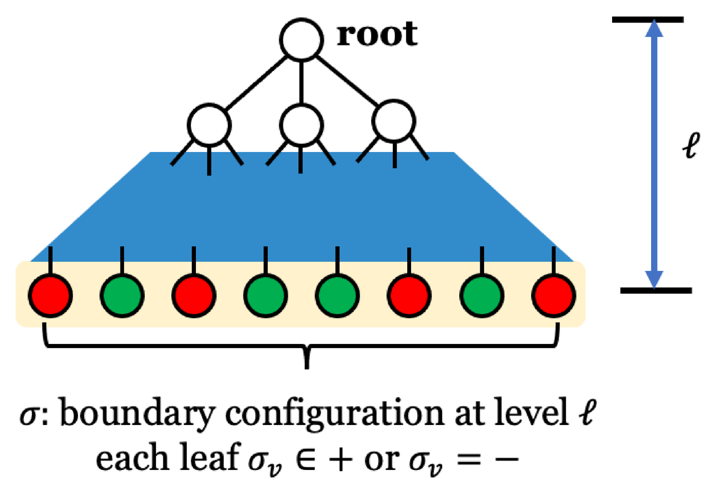
$$\forall \text{ independent set } \sigma \in \Omega, \quad \mu(\sigma) = \frac{\lambda^{|\sigma|_+}}{Z}, \text{ where } Z = \sum_{\sigma\in\Omega} \lambda^{|\sigma|_+}.$$



Phase transition

Uniqueness threshold $\lambda_c(\Delta) = \frac{(\Delta-1)^{(\Delta-1)}}{(\Delta-2)^{\Delta}} \approx \frac{e}{\Delta}$.

1 Phase transition in physics



conditional probability
$P^{\sigma}_{\text{root}} = \Pr[X(\text{root}) = + \mid \sigma]$

- if $\lambda < \lambda_c$, p^{σ}_{root} is independent with σ when $\ell \to \infty$;
- if $\lambda > \lambda_c$, p^{σ}_{root} is correlated with σ for any ℓ.

2 Phase transition in computational complexity

- if $\lambda < (1-\delta)\lambda_c(\Delta)$, sampling algorithm with running time $n^{O((\log\Delta)/\delta)}$ [Weitz06];
- if $\lambda > \lambda_c(\Delta)$, sampling problem is intractable unless $\mathbf{NP} = \mathbf{RP}$ [Sly10].

3 Open problem

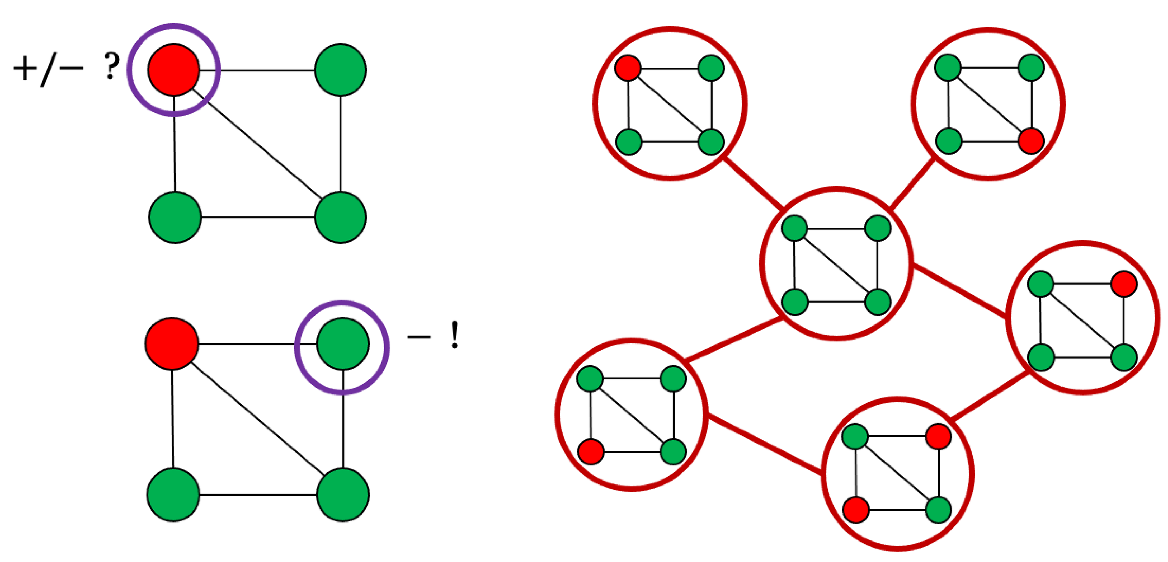
Sampling algorithm with running time $C(\delta)\cdot\mathrm{poly}(n)$ for all hardcore models satisfying $\lambda \le (1-\delta)\lambda_c(\Delta)$ and Δ can be **unbounded**.

Glauber dynamics

The Glauber dynamics

Start from an arbitrary independent set $X \in \Omega$;
For each update step:

1. pick $v \in V$ uniformly at random;
2. if all neighbours u of v satisfy $X_u = -$
$$X_v \leftarrow \begin{cases} + & \text{with probability } \lambda/(1+\lambda), \\ - & \text{with probability } 1/(1+\lambda); \end{cases}$$
3. if some neighbour u of v satisfies $X_u = +$, then $X_v \leftarrow -$.



mixing time : $T_{\text{mix}} = \max_{X_0} \min\{t \mid d_{\text{tv}}(X_t, \mu) \le 0.001\}$.

total variation distance : $d_{\text{tv}}(X_t, \mu) = \frac{1}{2}\sum_{\sigma\in\Omega} |\Pr[X_t = \sigma] - \mu(\sigma)|$.

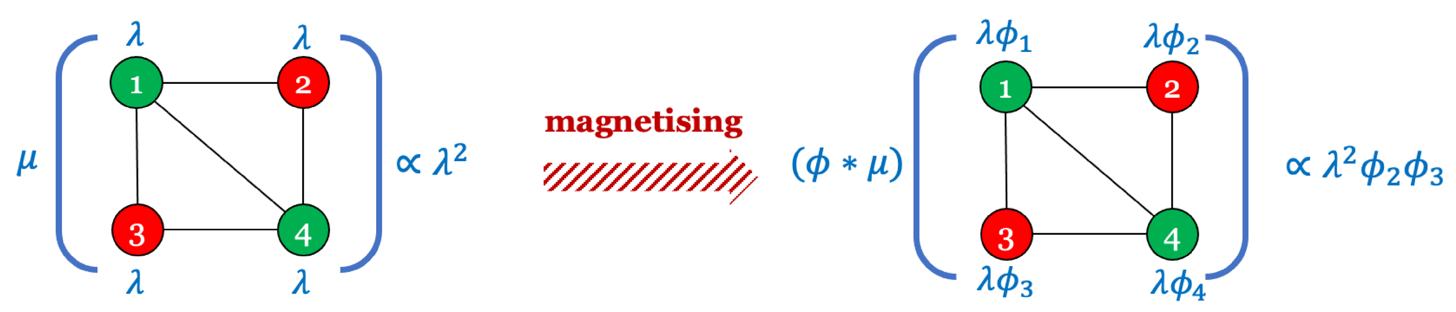
Our results

Mixing time for hardcore model when $\lambda \le (1-\delta)\lambda_c$

Anari, Liu and Oveis Gharan, 2020	$n^{O(1/\delta)}$
Chen, Liu and Vigoda, 2021	$\Delta^{O(\Delta^2/\delta)} \cdot n\log n$
This work, 2021	$e^{O(1/\delta)} \cdot n^2\log n$
Our follow-up work, 2022	$e^{O(1/\delta)} \cdot n\log n$
Chen and Eldan, 2022	$e^{O(1/\delta)} \cdot n\log n$

Results for general distributions μ over $\{-,+\}^V$

- *Influence matrix*: $\Psi(u,v) = |\Pr_\mu[v = + \mid u = +] - \Pr_\mu[v = + \mid u = -]|$.
- *Spectral independence (SI)*: for any conditional distribution induced by μ, the spectral radius $\rho(\Psi) \le C$.
- *Magnetising joint distribution with local fields $\phi = (\phi_v)_{v\in V} \in \mathbb{R}^V_{>0}$*:
$$(\phi * \mu)(\sigma) \propto \mu(\sigma) \prod_{v\in V:\sigma_v=+} \phi_v.$$

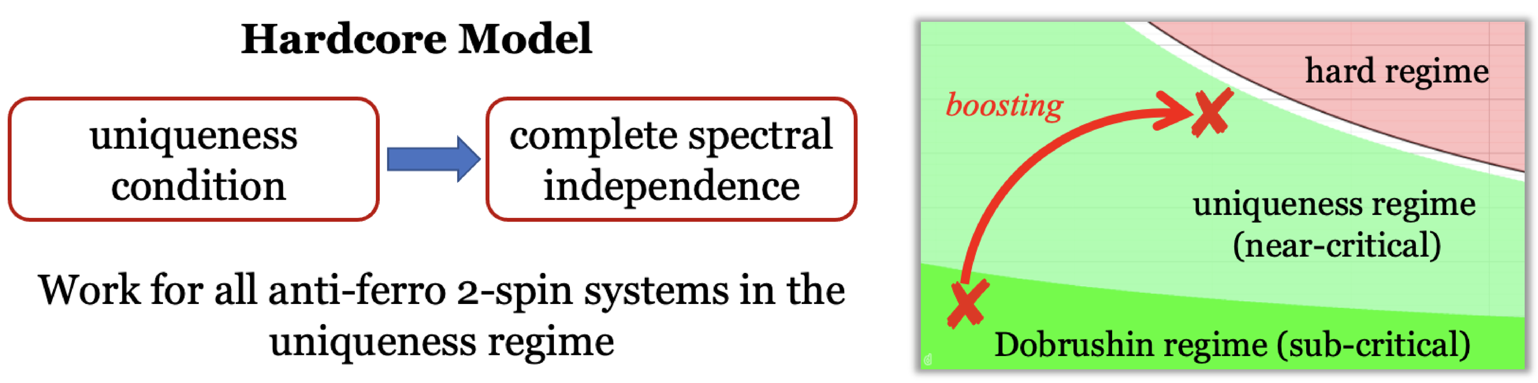


- *Complete SI*: $\forall\phi \in (0,1]^V$, $(\phi * \mu)$ is spectrally independent.
- *Spectral gap*: $1 - \lambda_2$, where λ_2 is the second largest eigenvalue of the transition matrix P of the Glauber dynamics on μ:
$$T_{\text{mix}} = O\left(\frac{1}{\lambda_{\text{gap}}}\log\frac{1}{\mu_{\min}}\right), \text{ where } \mu_{\min} = \min_{\sigma\in\Omega}\mu(\sigma).$$

Boosting result for spectral gap of Glauber dynamics

For any C-completely SI distribution μ, any $\theta \in (0,1)$
$$\lambda_{\text{gap}}(\mu) \ge \left(\frac{\theta}{2}\right)^{2C+7} \lambda^*_{\text{gap}}(\theta * \mu), \quad \text{where } \theta_v = \theta \text{ for all } v \in V.$$

- $\lambda_{\text{gap}}(\mu)$: the spectral gap of the Glauber dynamics on μ
- $\lambda^*_{\text{gap}}(\theta * \mu)$: the minimum spectral gap of the Glauber dynamics on conditional distributions induced by $(\theta * \mu)$.



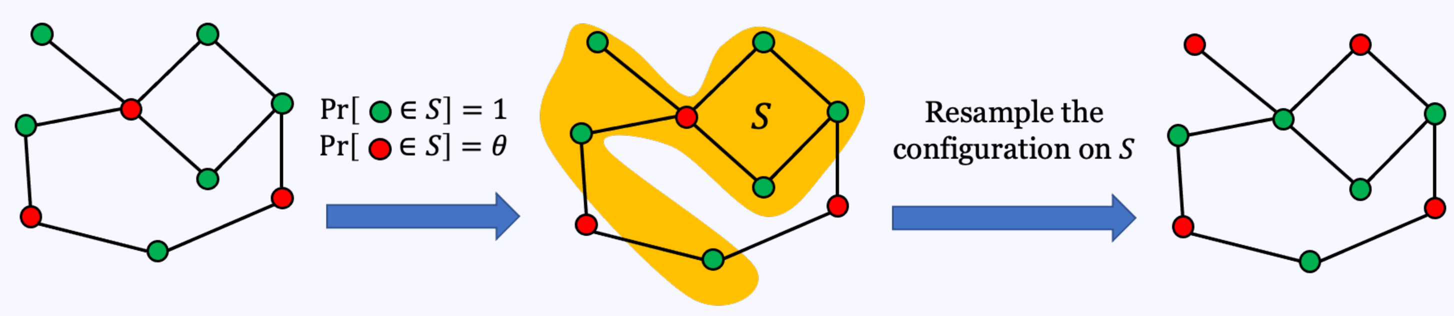
Field dynamics

The Field dynamics

Boolean distribution μ over $\{-,+\}^V$ and parameter $\theta \in (0,1)$.

Start from an arbitrary feasible state $X \in \Omega$;
For each update step:

1. construct S by selecting each $v \in V$ independently with probability
$$p_v = \begin{cases} 1 & \text{if } X_v = -, \\ \theta & \text{if } X_v = +; \end{cases}$$
2. resample $X_S \sim (\theta * \mu)_S(\cdot \mid X_{V\setminus S})$.



- **Comparison lemma**: $\lambda_{\text{gap}}(\mu) \ge \lambda^{\text{Field}}_{\text{gap}}(\theta,\mu)\lambda^*_{\text{gap}}(\theta * \mu)$.
- **Mixing lemma**: If μ is C-Completely SI, then $\lambda^{\text{Field}}_{\text{gap}}(\theta,\mu) \ge \left(\frac{\theta}{2}\right)^{2C+7}$.