# Perfect sampling from spatial mixing

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Joint work with

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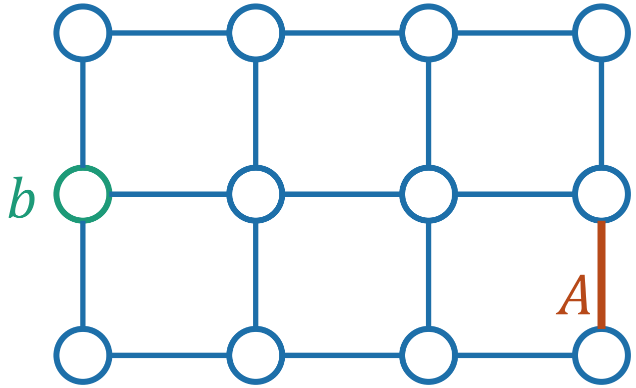
# Spin systems and Gibbs distributions

finite graph $G = (V, E)$



Parameters

vertex: random variable in $[q] = \{0,1, \dots, q-1\}$

**external field**: vector $b \in \mathbb{R}^q_{\geq 0}$ in each vertex

**interaction**: symmetric matrix $A \in \mathbb{R}^{q\times q}_{\geq 0}$ on each edge

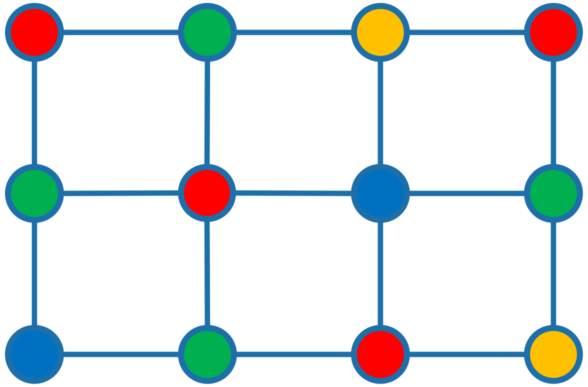
$$\forall \underset{\uparrow}{\sigma} \in [q]^V, \qquad \underset{\uparrow}{\mu}(\sigma) \propto \underbrace{\prod_{v\in V} b(\sigma_v) \prod_{e=\{u,v\}\in E} A(\sigma_u, \sigma_v)}_{\uparrow}$$

configuration

Gibbs distribution

weight $w(\sigma)$

# Example: graph coloring



**proper $\boldsymbol{q}$-colouring in graph $\boldsymbol{G=(V,E)}$**

- Each vertex $v \in V$ take a colour $\sigma_v \in [q]$
- Each edge $\{u, v\} \in E$ is not monochromatic

$$\sigma_u \neq \sigma_v$$

$$b = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

constant vector

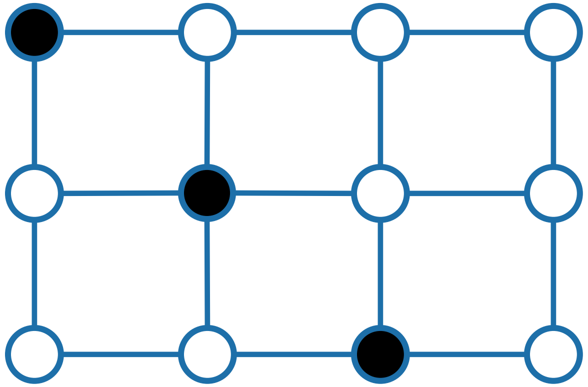
$$A = \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & 0 & \vdots \\ 1 & \cdots & 0 \end{bmatrix}$$

$$A(i,i) = 0$$
$$A(i,j) = 1$$

$$\mu(\sigma) \propto \begin{cases} 1 & \text{if } \sigma \text{ is a proper colouring} \\ 0 & \text{otherwise} \end{cases}$$

**uniform distribution** over all proper $q$-colourings in $G$

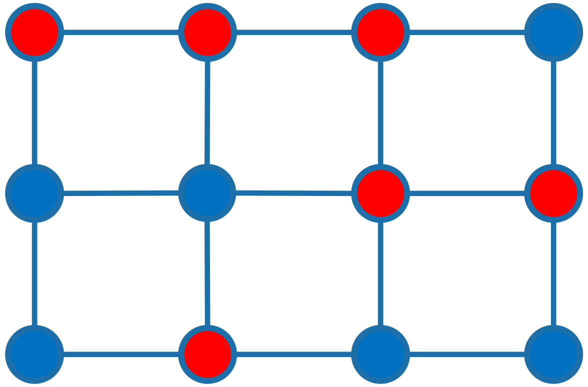
# Examples: hardcore model and Ising model



**Hardcore model in $G$ with parameter $\lambda > 0$**

$\forall \sigma \in \{0,1\}^V$ s.t. $S_\sigma = \{v \in V \mid \sigma_v = 1\}$ is an independent set

$$\mu(\sigma) \propto \lambda^{|S_\sigma|}$$



**Ising model in $G$ with parameter $\beta > 0$**

$$\forall \sigma \in \{0,1\}^V, \mu(\sigma) \propto \beta^{m(\sigma)}$$

$m(\sigma) = |\{\{u,v\} \in E \mid \sigma_u = \sigma_v\}|$ is #monochromatic edges

# Sampling problem for spin systems

**Input**: ① a graph $G = (V, E)$, vector $b \in \mathbb{R}^{q}_{\geq 0}$, symmetric matrix $A \in \mathbb{R}^{q \times q}_{\geq 0}$

specify ***Gibbs distribution*** $\mu(\sigma) \propto \prod_{v \in V} b(\sigma_v) \prod_{e=\{u,v\} \in E} A(\sigma_u, \sigma_v)$

② an error bound $\epsilon \geq 0$

**Output:** a random sample $X \in [q]^V$ such that

$$d_{TV}(X, \mu) = \frac{1}{2} \sum_{\sigma \in [q]^V} |\Pr[X = \sigma] - \mu(\sigma)| \leq \epsilon$$

- ***Approximate*** sampling problem: return random samples with **bounded error** $\epsilon > 0 \left(\epsilon = \frac{1}{\mathrm{poly}(n)}\right)$
- ***Perfect*** sampling problem: return random samples **without error** $\epsilon = 0$

# Sampling problem for spin systems

**Input**: ① a *sub-exp growth* graph $G = (V, E)$, vector $b \in \mathbb{R}_{\geq 0}^{q}$, symmetric matrix $A \in \mathbb{R}_{\geq 0}^{q \times q}$

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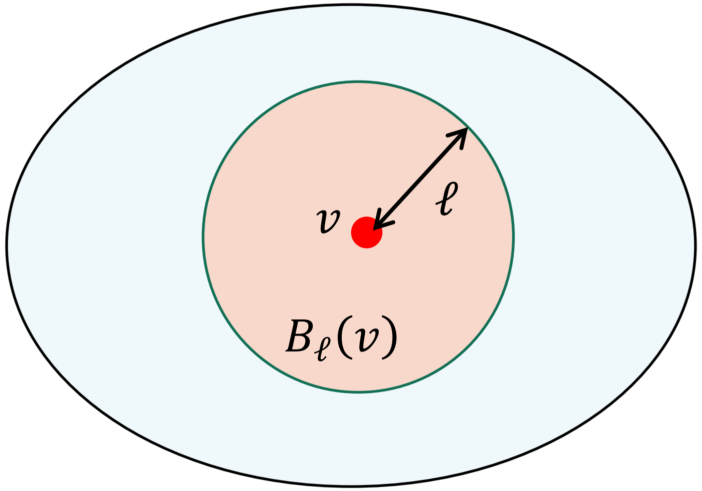
## Sub-exp growth graph

A family of graphs $\mathcal{G}$ has sub-exp growth if

$\exists$ a **sub-exp** function $s: \mathbb{N} \to \mathbb{N}$ s.t. $\forall G = (V, E) \in \mathcal{G}$,

$$\forall v \in V, \ell \in \mathbb{N}, \quad |B_\ell(v)| \leq \boldsymbol{s(\ell) = \exp(o(\ell))}$$

$B_\ell(v) = \{ u \in V \mid \mathrm{dist}_G(v, u) \leq \ell \}$ ***ball*** of radius $\ell$ centred at $v$



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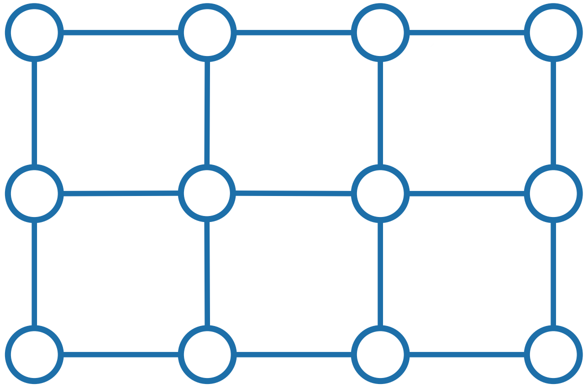
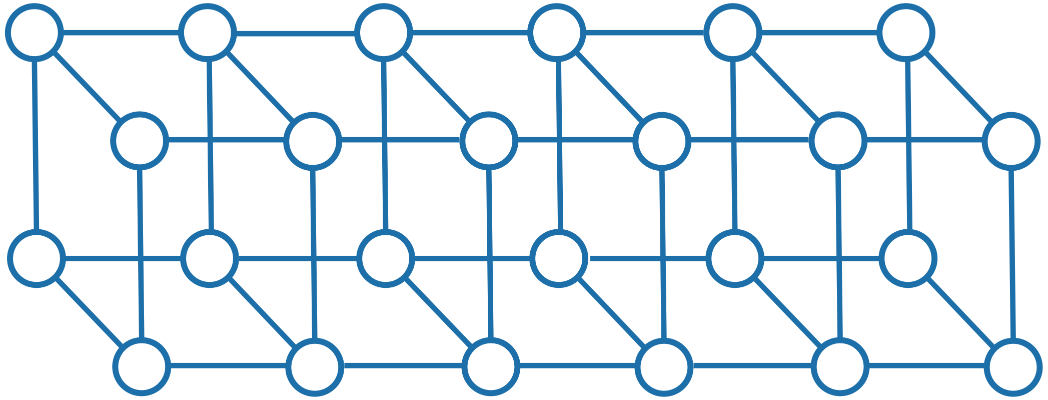
**Example**: let $d \in \mathbb{N}$, any **finite sub-graph** $G = (V, E)$ of $\mathbb{Z}^d$ has sub-exp growth

$$\forall v \in V, \ell \in \mathbb{N}, \qquad |B_\ell(v)| \leq (2\ell+1)^d = \mathrm{poly}(\ell) = \exp(o(\ell))$$





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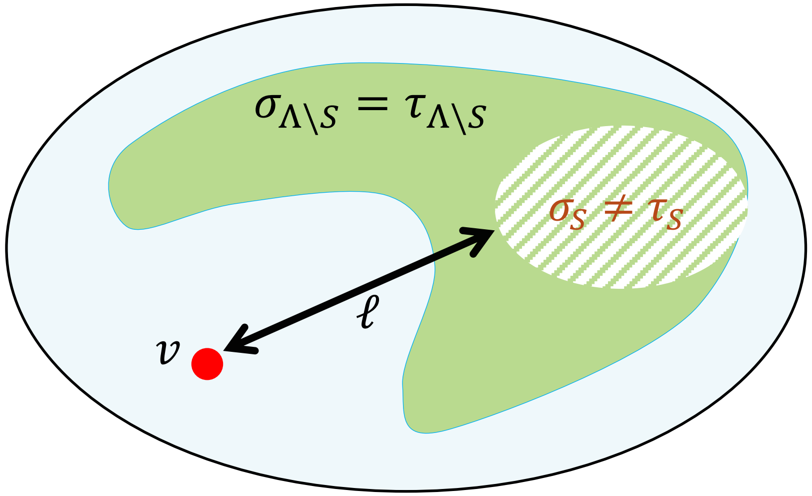
**Question:**

$\mu$ satisfies
**???** condition

→

**poly-time** sampling
algorithm exists

# Previous work: Strong spatial mixing v.s. approximate sampling



- Partial configurations $\sigma, \tau \in [q]^\Lambda$ on $\Lambda \subseteq V$

$$S = \{w \in \Lambda \mid \sigma_w \neq \tau_w\}$$

- Vertex $v \in V$ with distance to ***disagreement***

$$\ell = \min\{\mathrm{dist}_G(v, w) \mid w \in S\}$$

- Strong Spatial Mixing (***SSM***)

$$d_{TV}(\mu_v^\sigma, \mu_v^\tau) \leq \alpha \exp(-\beta\ell)$$

TV distance
Influence on $v$

marginals on $v$
conditional on $\sigma$ or $\tau$

exponential decay
$\alpha, \beta = \Theta(1)$

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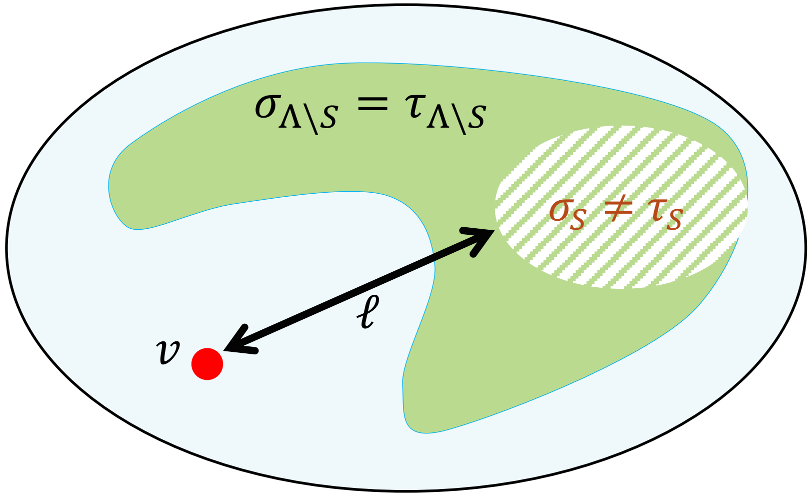

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***Some sufficient conditions*** for SSM on graph $G$ with max degree $\Delta$

- $q$-colouring: ($q > 2\Delta$) or (triangle-free and $q > 1.763\Delta$)
- hardcore: $\lambda < \lambda_c(\Delta) = \frac{(\Delta-1)^{(\Delta-1)}}{(\Delta-2)^\Delta} \approx \frac{e}{\Delta}$
- Ising model: $\frac{\Delta-2}{\Delta} < \beta < \frac{\Delta}{\Delta-2}$

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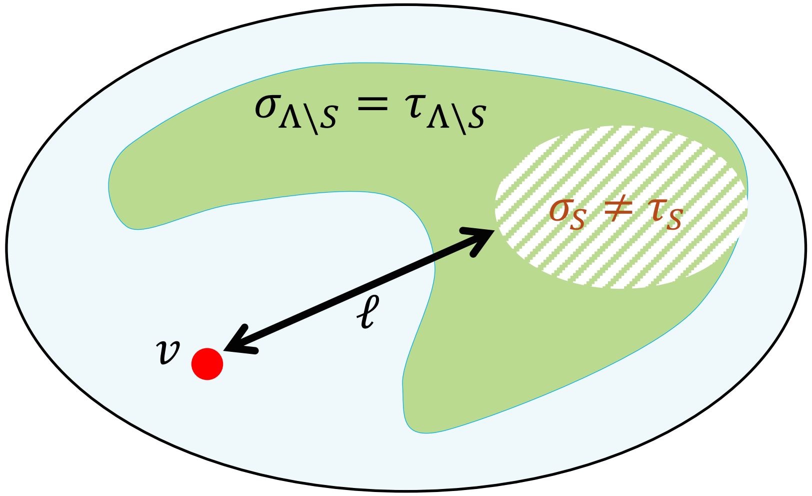

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**Theorem** [Dyer, Sinclair, Vigoda and Weitz 2004]

SSM

sub-exp growth

near-liner time
approximate sampler

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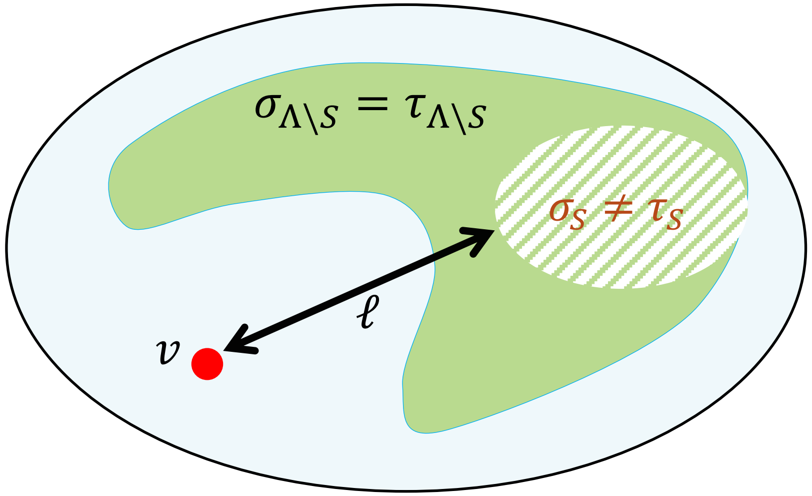

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SSM

sub-exp growth

Markov chain optimal mixing

# Our results

SSM

**sub-exp growth graph**

linear time *perfect* sampler

**Main result** [This work]

Constants $q \in \mathbb{N}, b \in \mathbb{R}_{\geq 0}^{q}$ and $A \in \mathbb{R}_{\geq 0}^{q \times q}$ . There exists an algorithm such that

- given any graph $G = (V, E)$ with **sub-exp growth**
- output a **perfect sample** from Gibbs distribution $\mu$ in time $O(n)$, $n = |V|$

**Remark**: the linear running time $C \cdot n$ of the algorithm

the ***constant*** $C = C(q, b, A, s)$ depends on

- number of spins $q$, external vector $b$ and interaction matrix $A$
- parameters in sub-exp function $s: \mathbb{N} \to \mathbb{N}$ (recall $|B_\ell(v)| \leq s(\ell) = \exp(o(\ell))$)

$C$ does ***not*** depend on $n$

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*Example*: if $G = (V, E)$ is a **finite subgraph of $\mathbb{Z}^d$**, then the constant

$$C = C(q, A, b, d)$$

# Applications for spin systems

| Model | Graph | Parameters |
|---|---|---|
| $q$-colouring | sub-exp growth | $q > 2\Delta$ |
| $q$-colouring | sub-exp growth<br>triangle-free | $q > 1.763\Delta$ |
| hardcore | sub-exp growth | $\lambda < \lambda_c(\Delta) \approx \frac{e}{\Delta}$ |
| Ising | sub-exp growth | $1 - \frac{2}{\Delta} < \beta < 1 + \frac{2}{\Delta}$ |

Our algorithm also works for **general graphs**, but with a **stronger SSM condition**

***Example***: $q$-colouring on general graphs

- our condition: $q \geq \Delta^2 - \Delta + 2$
- state-of-the-art: $q = \Omega(\Delta)$ [Jain, Sah, Sawhney, 2021] [Liu, Sinclair, Srivastava, 2019]

# Other techniques & our technical contribution

| **Techniques** | **Graph** | **#Colours** | **Running time** |
|---|---|---|---|
| Reduction to deterministic approximate counting [JVV86,LSS19] | general | $q > 2\Delta$ | $n^{\mathrm{poly}(\Delta)}$ |

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| Depth-First-Sampling [Anand, Jerrum, 2021] | sub-exp growth | $q > 2\Delta$<br>$q > 1.763\Delta$ (triangle-free) | $O_{q,\Delta}(n)$ |

The Depth-First-Sampling algorithm could recover our main result

SSM

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| ***Bayes filter* [This paper, 2021]** | sub-exp growth | $q > 2\Delta$<br>$q > 1.763\Delta$ (triangle-free) | $O_{q,\Delta}(n)$ |

**same principle**

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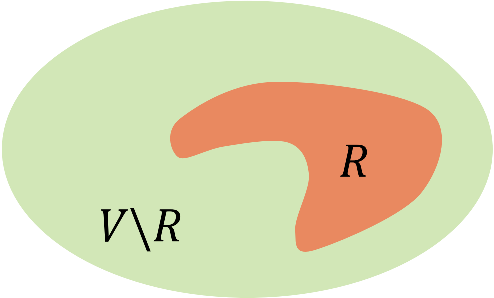
| **MCMC approximate sampler**<br>[Chen, Delcourt, Moitra, Perarnau, 2019] | **general** | $q \geq \left(\frac{11}{6} - \epsilon_0\right)\Delta,\ \epsilon_0 \geq 10^{-5}$ | $\tilde{O}_{q,\Delta}(n)$ |

**Open problem**: close the ***gap*** between perfect sampling and approximate sampling

# The perfect sampling algorithm

Maintains a random pair $(X, R)$, where $X \in [q]^V$ and $R \subseteq V$ s.t.

$(X, R)$ satisfies the ***conditional Gibbs property***



**Conditional Gibbs property** [Huber, Fill, 2000; Guo, Jerrum, Liu, 2017; Feng, Vishnoi, Yin 2019 ]

For any $\Lambda \subseteq V$, any $\sigma \in [q]^\Lambda$, conditional on $R = \Lambda$ and $X_R = \sigma$, $X_{V\setminus R} \sim \mu_{V\setminus R}^{\sigma}$:

$$\forall \tau \in [q]^{V\setminus R},\ \Pr_{(X,R)}\left[X_{V\setminus R} = \tau \mid R = \Lambda \wedge X_R = \sigma\right] = \mu_{V\setminus R}^{\sigma}(\tau)$$

**Remarks about Conditional Gibbs property**

The distribution of $(X_R, R)$ can be ***arbitrary***, but $X_{V\setminus R}$ must follow $\boldsymbol{\mu_{V\setminus R}^{X_R}}$ if $(X_R, R)$ is given.

- $R$ is the set of "**bad variables**" and $V\setminus R$ is the set of "**good variables**"
- In general, the distribution of $(X, R)$ is **not unique**
- If $R = V$, then $X$ is arbitrary; if $R = \emptyset$, then $X \sim \mu$

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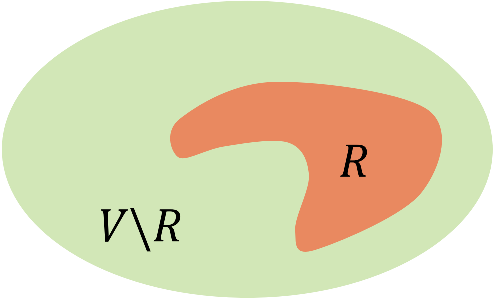

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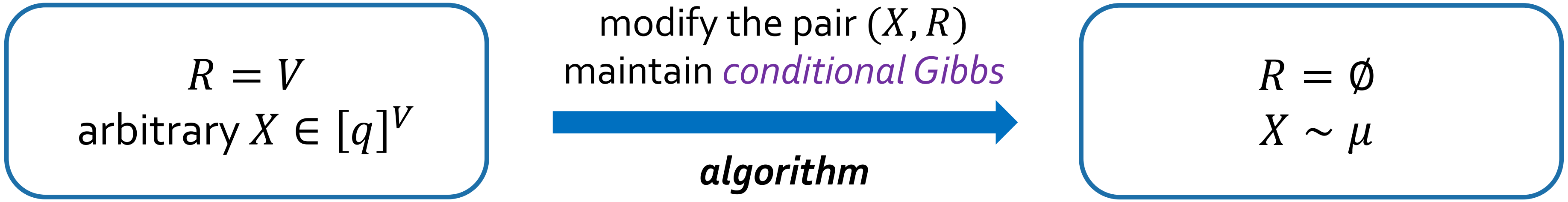


$R = V$
arbitrary $X \in [q]^V$

modify the pair $(X, R)$
maintain *conditional Gibbs*

***algorithm***

$R = \emptyset$
$X \sim \mu$

# Warm-up: A simple case

**Input**: Gibbs distribution $\mu$ in $G = (V, E)$ (e.g., uniform distribution over $q$-colourings) random pair $(X, R)$ such that

- $R = \{u\}$ and $X_u =$ Red
- $X_{V\setminus\{u\}} \sim \mu_{V\setminus\{u\}}(\cdot \mid u \text{ is Red})$

} **conditional Gibbs property** ✓

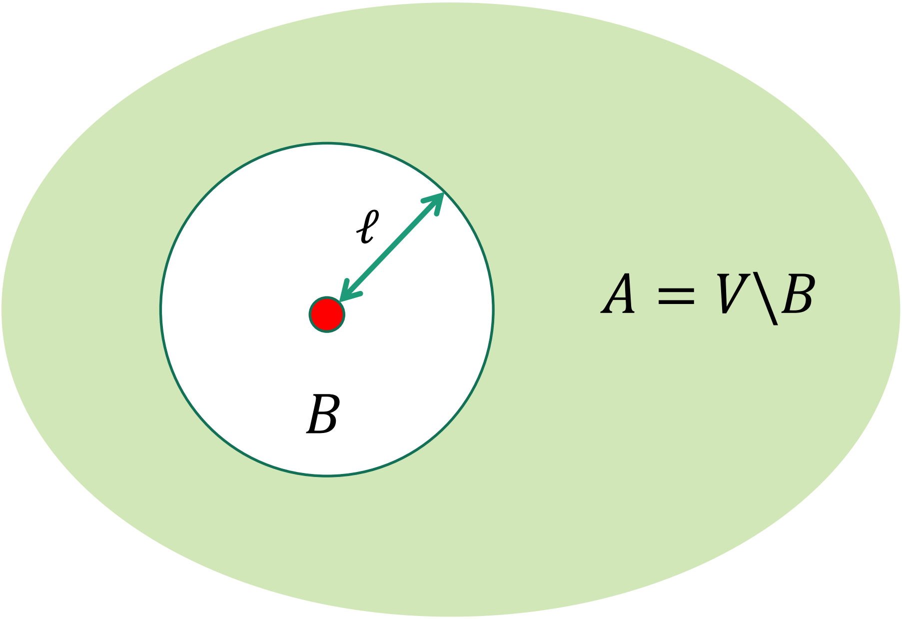
**Idealised goal:** modify such $(X, R = \{u\})$ so that $R = \emptyset$ and $X \sim \mu$

$B$: $\ell$-ball centred at $u$, $\ell = O(1)$; $A = V\setminus B$

Input: $X_A \sim \mu_A(\cdot \mid u \text{ is Red})$

Idealised goal: $X_A \sim \mu_A(\cdot) = \sum_{c\in[q]} \mu_u(c)\mu_A(\cdot \mid u \text{ is } c)$

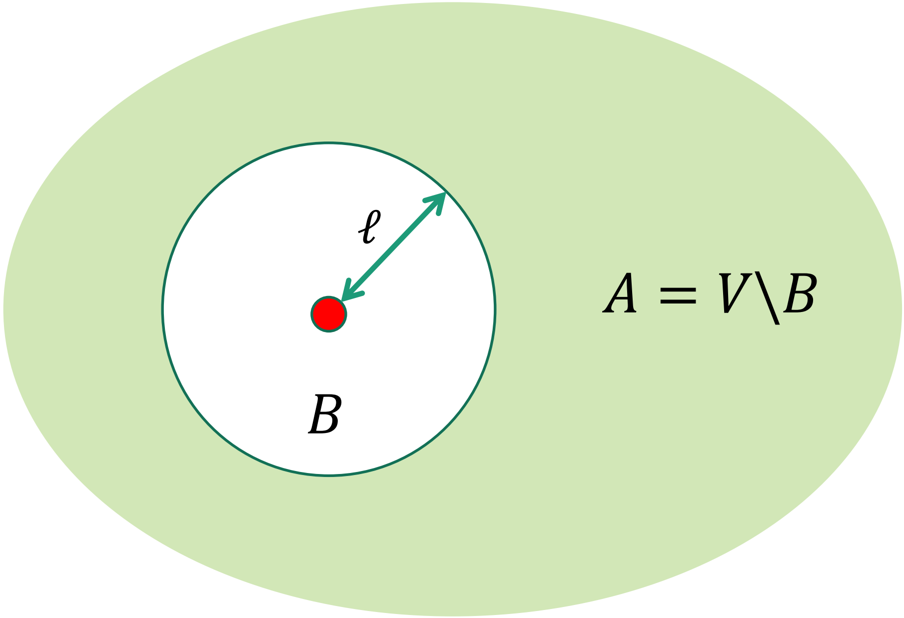
SSM → $\mu_A(\cdot \mid u \text{ is Red}) \approx \mu_A(\cdot)$



SSM → $\mu_A(\cdot \mid u \text{ is Red}) \approx \mu_A(\cdot)$

$X_A \sim \mu_A(\cdot \mid u \text{ is Red}) \xrightarrow{\textit{filter}} X_A \sim \mu_A(\cdot)$

Idea: use a ***filter*** to transform the distribution



$$\forall \sigma \in [q]^A, \qquad \Pr[X_A \text{ passes filter} \mid X_A = \sigma] \propto \frac{\mu_A(\sigma)}{\mu_A(\sigma \mid u \text{ is Red})}$$

target distribution

input distribution

$$\Pr[X_A = \sigma \wedge X_A \text{ passes filter}] \propto \mu_A(\sigma \mid u \text{ is Red}) \cdot \frac{\mu_A(\sigma)}{\mu_A(\sigma \mid u \text{ is Red})} = \mu_A(\sigma)$$

$$\Longrightarrow \quad \Pr[X_A = \sigma \mid X_A \text{ passes filter}] = \mu_A(\sigma)$$

If $X_A$ passes the filter, then $X_A \sim \mu_A$

SSM $\rightarrow$ $\mu_A(\cdot \mid u \text{ is Red}) \approx \mu_A(\cdot)$

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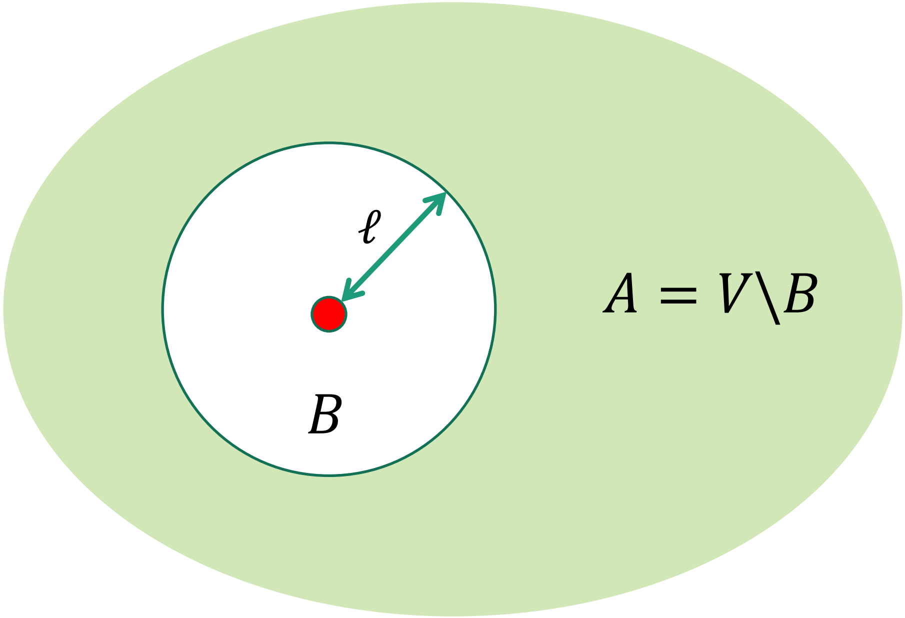


$$\forall \sigma \in [q]^A, \qquad \Pr[X_A \text{ passes filter} \mid X_A = \sigma] \propto \frac{\mu_A(\sigma)}{\mu_A(\sigma \mid u \text{ is Red})} \begin{array}{l} \longleftarrow \text{target distribution} \\ \longleftarrow \text{input distribution} \end{array}$$

*(by Bayes' Law)*

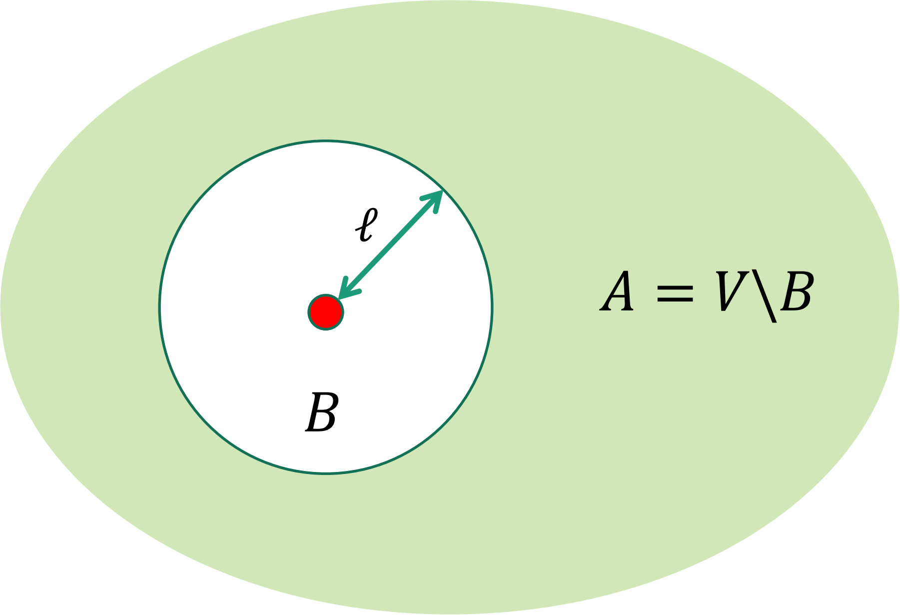
$$\mu_A(\sigma \mid u \text{ is Red}) = \frac{\mu_A(u \text{ is Red} \mid A \leftarrow \sigma)\mu_A(\sigma)}{\mu_u(\text{Red})}$$

the event $A \leftarrow \sigma$:
vertices in $A$ are coloured as $\sigma$

SSM $\Rightarrow$ $\mu_A(\cdot \mid u \text{ is } \color{red}{\text{Red}}) \approx \mu_A(\cdot)$

$X_A \sim \mu_A(\cdot \mid u \text{ is } \color{red}{\text{Red}})$ $\xrightarrow{\textit{filter}}$ $X_A \sim \mu_A(\cdot)$

Idea: use a ***filter*** to transform the distribution



$$\forall \sigma \in [q]^A, \qquad \Pr[X_A \text{ passes filter} \mid X_A = \sigma] \propto \frac{\mu_A(\sigma)}{\mu_A(\sigma \mid u \text{ is } \color{red}{\text{Red}})}$$

target distribution

input distribution

*(by Bayes’ Law)*

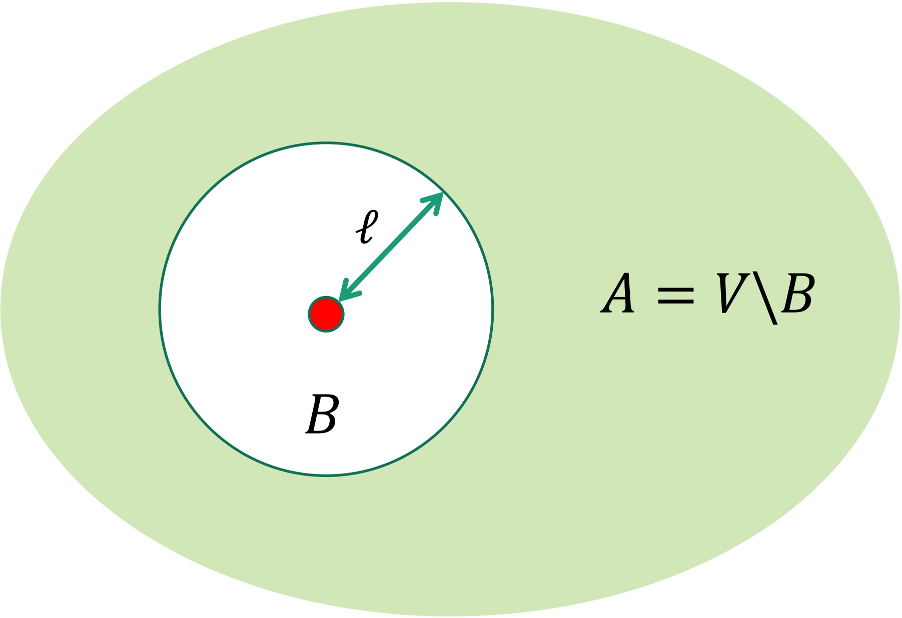
$$= \frac{\mu_A(\sigma)\mu_u(\color{red}{\text{Red}})}{\mu_u(\color{red}{\text{Red}} \mid A \leftarrow \sigma)\mu_A(\sigma)}$$

the event $A \leftarrow \sigma$:
vertices in $A$ are coloured as $\sigma$

$$\text{SSM} \longrightarrow \mu_A(\cdot \mid u \text{ is } \text{Red}) \approx \mu_A(\cdot)$$

$$X_A \sim \mu_A(\cdot \mid u \text{ is } \text{Red}) \xrightarrow{\textit{filter}} X_A \sim \mu_A(\cdot)$$

Idea: use a ***filter*** to transform the distribution



$$\forall \sigma \in [q]^A, \qquad \Pr[X_A \text{ passes filter} \mid X_A = \sigma] \propto \frac{\mu_A(\sigma)}{\mu_A(\,\sigma \mid u \text{ is Red}\,)}$$

target distribution

input distribution

*(by Bayes' Law)*

$$= \frac{\mu_A(\sigma)\mu_u(\text{Red})}{\mu_u(\,\text{Red} \mid A \leftarrow \sigma\,)\mu_A(\sigma)}$$

the event $A \leftarrow \sigma$:
vertices in $A$ are coloured as $\sigma$

*(cancel $\mu_A(\sigma)$)*

$$= \frac{\mu_u(\text{Red})}{\mu_u(\text{Red} \mid A \leftarrow \sigma)} \propto \frac{1}{\mu_u(\text{Red} \mid A \leftarrow \sigma)}$$

*($\mu_u(\text{Red})$ is independent with $\sigma$)*

$$\forall \sigma \in [q]^A, \qquad \Pr[X_A \text{ passes filter} \mid X_A = \sigma] \propto \frac{1}{\mu_u(\text{Red} \mid A \leftarrow \sigma)} = \frac{1}{\mu_u(\text{Red} \mid \partial B \leftarrow \sigma_{\partial B})}$$

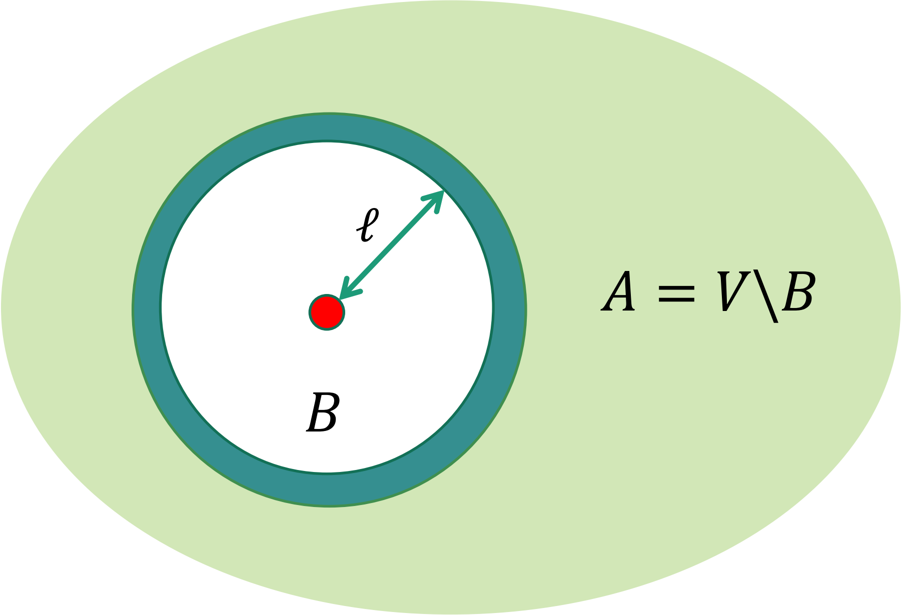
*(by conditional independence)*

$\partial B \subseteq A$ is the ***outside boundary*** of $B$

$$\partial B = \{w \notin B \mid \exists w' \in B \text{ st } \{w, w'\} \in E\}$$

$\partial B$ ***separates*** $B$ and $A \setminus \partial B$ in graph $G$

➡ <u>conditional independence</u>



$$\forall \sigma \in [q]^A, \qquad \Pr[X_A \text{ passes filter} \mid X_A = \sigma] \propto \frac{1}{\mu_u(\text{Red} \mid A \leftarrow \sigma)} = \frac{1}{\mu_u(\text{Red} \mid \partial B \leftarrow \sigma_{\partial B})}$$

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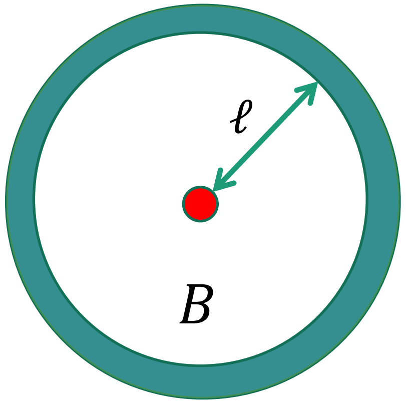
➡ conditional independence

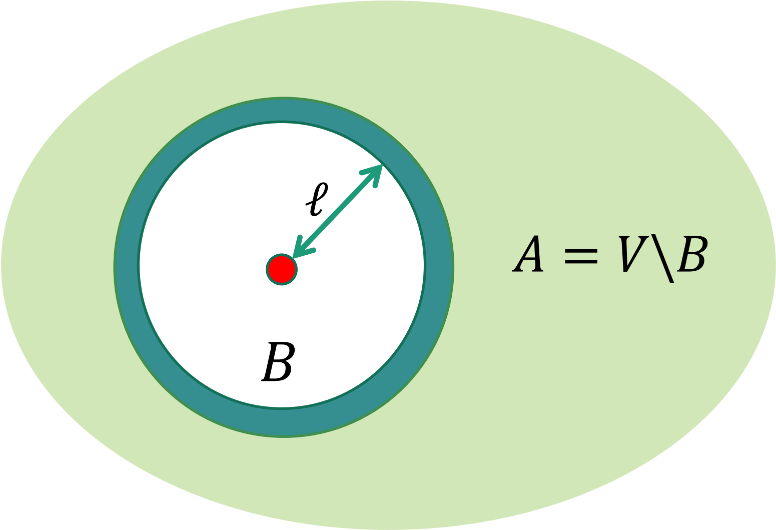


$\ell = O(1)$

$|B| = O(1)$

$$\forall \sigma \in [q]^A, \qquad \Pr[X_A \text{ passes filter} \mid X_A = \sigma] = \frac{\min_{\tau \in [q]^{\partial B}} \mu_u(\text{Red} \mid \partial B \leftarrow \tau)}{\mu_u(\text{Red} \mid \partial B \leftarrow \sigma_{\partial B})} \leq 1$$

# Bayes filter: simple case



- Reveal the configuration $X_{\partial B}$
- flip a coin such that

(Bayes filter)

$$\Pr[\mathrm{HEADS}] = \frac{\min_{\tau \in [q]^{\partial B}} \mu_u(\mathrm{Red} \mid \partial B \leftarrow \tau)}{\mu_u(\mathrm{Red} \mid \partial B \leftarrow X_{\partial B})}$$

- **If** the outcome is HEADS, **then**
  - resample $X_B \sim \mu_B(\cdot \mid X_{\partial B})$
  - **Return** the pair $(X, \emptyset)$

(pass the Bayes filter, then $X_A \sim \mu_A$)

$(X_B \sim \mu_B^{X_A} = \mu_B^{X_{\partial B}}$, then $X \sim \mu)$

- **If** the outcome is not HEAD, **then**
  - **Return** the pair $(X, \underbrace{\{u\} \cup \partial B}_{R})$

(not pass the Bayes filter)

(Bayes filter only reveal $X_{\partial B}$, $R = \{u\} \cup \partial B$)

# General case

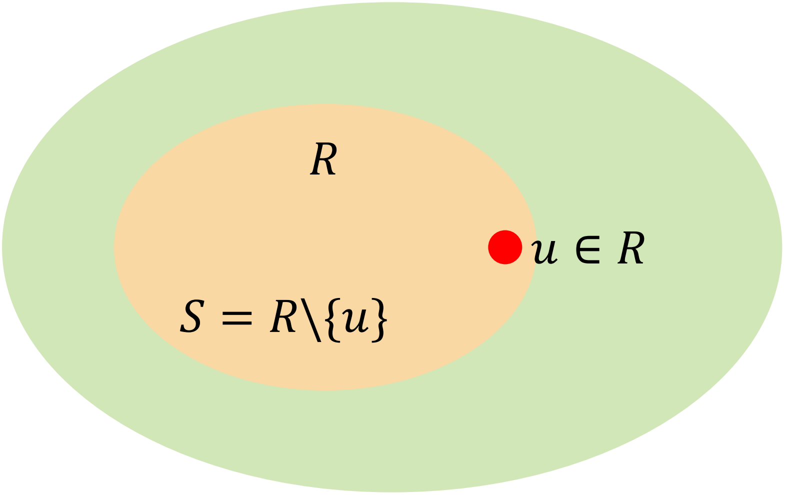
arbitrary $(X, R)$
(conditional Gibbs)

***algorithm***

new $(X, R)$
(conditional Gibbs)

size of $R$ decreases in expectation $V \backslash R$

- Reveal $X$ on $R$, say $X_R = \sigma$;
- Pick the vertex $u \in R$ with the lowest index
- Define set $S = R \backslash \{u\}$



# General case

arbitrary $(X, R)$
(conditional Gibbs)

***algorithm***

new $(X, R)$
(conditional Gibbs)

size of $R$ decreases in expectation

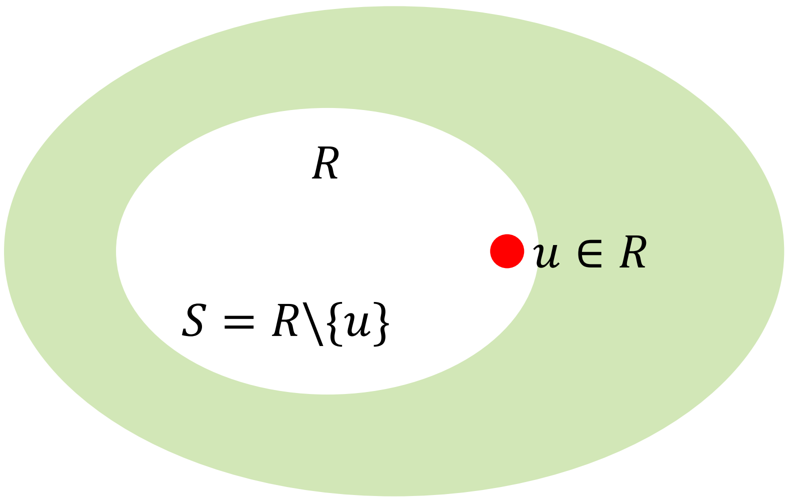
- Reveal $X$ on $R$, say $X_R = \sigma$;
- Pick the vertex $u \in R$ with the lowest index
- Define set $S = R \backslash \{u\}$
- Consider the Gibbs distribution

$$\pi = \mu_{V \backslash S}(\cdot \mid S \leftarrow \sigma_S)$$

- $(X_{V \backslash S}, \{u\})$ satisfies conditional Gibbs w.r.t. $\pi$

$(\{u\} = R \backslash S)$ $X_{V \backslash \{u\}} \sim \pi_{V \backslash \{u\}}(\cdot \mid u \leftarrow \sigma_u)$

- Back to the ***simple case*** that $R$ only contains 1 vertex

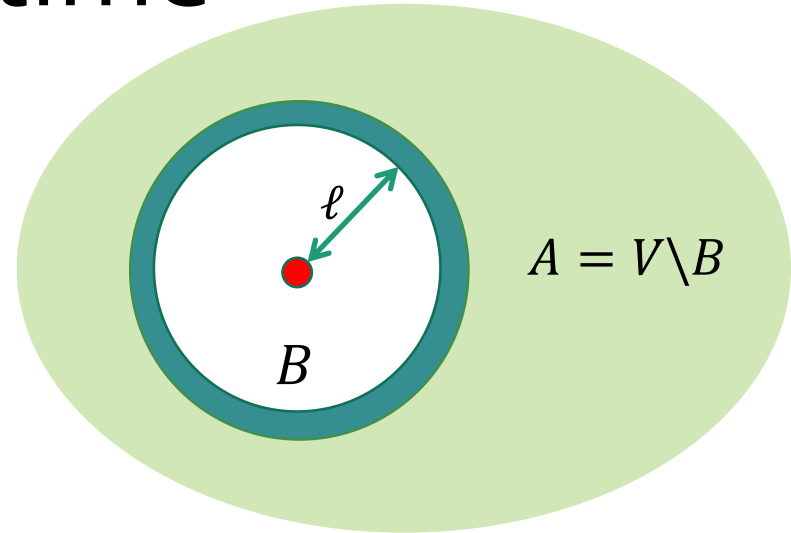


# Analysis of running time

**Bayes Filter**

$$\Pr[\text{HEADS}] = \frac{\mu_{\min}}{\mu_u(X_u \mid X_{\partial B})} = \frac{\mu_u(X_u \mid \partial B \leftarrow \tau^*)}{\mu_u(X_u \mid \partial B \leftarrow X_{\partial B})}$$

$\mu_{\min} = \min\{\mu_u(X_u \mid \tau) \mid \tau \in [q]^{\partial B}\}$ is achieved by $\tau^*$



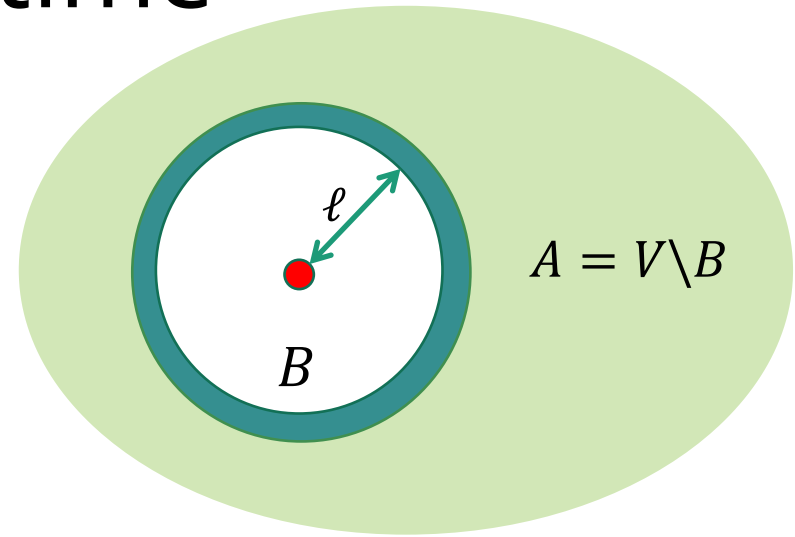
***SSM*** → [Spinka 2020] → $\Pr[\text{HEADS}] = 1 - \exp(-\Omega(\ell))$

**Pass** the Bayes filter → **Eliminate** vertex $u$ from $R$

**Not pass** the Bayes filter → **Add** $\partial B$ into $R$, where $|\partial B| = \exp(o(\ell))$

Choose $\ell = O(1)$ such that the **size of $R$** **decays in expectation** in every step

# Analysis of running time



**Bayes Filter**

$$\Pr[\mathrm{HEADS}] = \frac{\mu_{\min}}{\mu_u(X_u \mid X_{\partial B})} = \frac{\mu_u(X_u \mid \partial B \leftarrow \tau^*)}{\mu_u(X_u \mid \partial B \leftarrow X_{\partial B})}$$

$\mu_{\min} = \min\left\{ \mu_u(X_u \mid \tau) \mid \tau \in [q]^{\partial B} \right\}$ is achieved by $\tau^*$

***SSM***

[Spinka 2020]

$\Pr[\mathrm{HEADS}] = 1 - \exp(-\Omega(\ell))$

**Pass** the Bayes filter

**Not pass** the Bayes filter

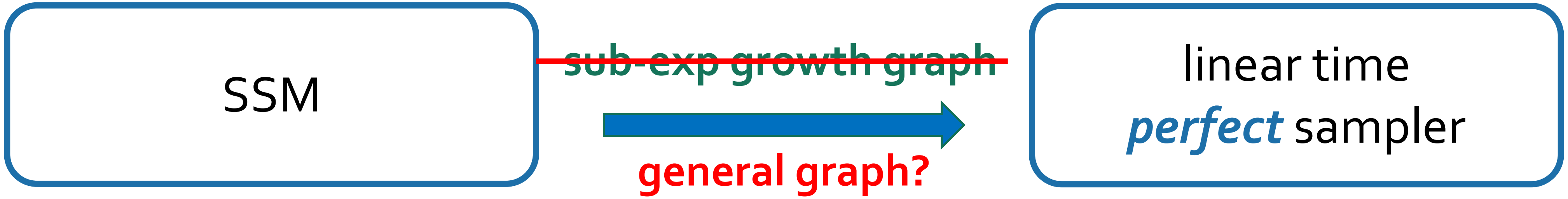
**Eliminate** vertex $u$ from $R$

**Add** $\partial B$ into $R$, where $|\partial B| = \exp(o(\ell))$

**Q**: Is the **Weak Spatial Mixing (WSM)** enough for this analysis?

**A**: **No**, in the general case, we do analysis on **conditional distributions**.

# Open Problems



SSM

~~sub-exp growth graph~~

**general graph?**

linear time

***perfect*** sampler

Uniform $q$-colourings on graph $G = (V, E)$ with max degree $\Delta$

- Perfect sampler when $q > 2\Delta$ with running time $\tilde{O}_{q,\Delta}(n)$ ??

Hardcore model on graph $G = (V, E)$ with parameter $\lambda > 0$ and max degree $\Delta$

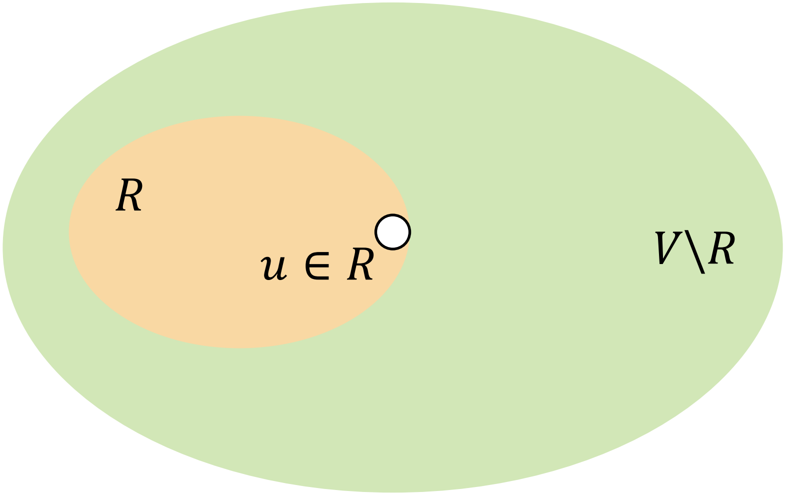
- perfect sampler when $\lambda_c < \lambda_c(\Delta)$ with running time $\tilde{O}_{\lambda,\Delta}(n)$??

**Thank you!** **Q&A**

# Appendix

# Unified Bayes filter

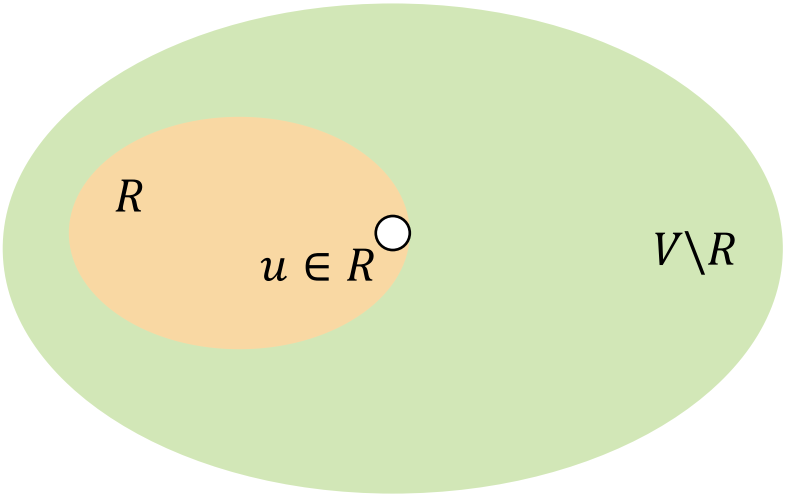
- Pick the vertex $u \in R$ with the lowest index

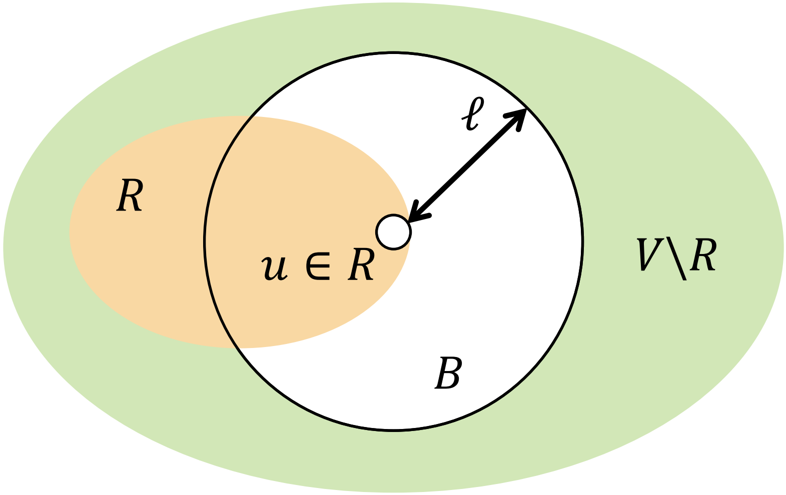


# Unified Bayes filter

- Pick the vertex $u \in R$ with the lowest index
- Define the regime

$$B = B_\ell(u) \backslash R \cup \{u\}$$





# Unified Bayes filter

- Pick the vertex $u \in R$ with the lowest index
- Define the regime

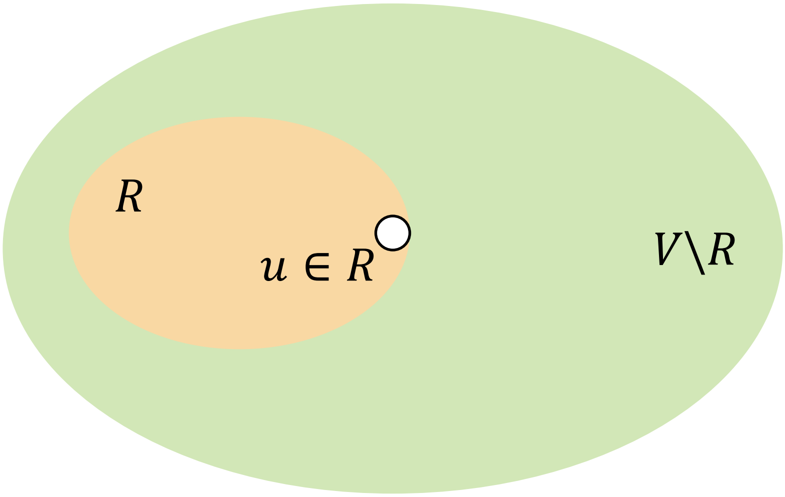
$$B = B_\ell(u) \backslash R \cup \{u\}$$

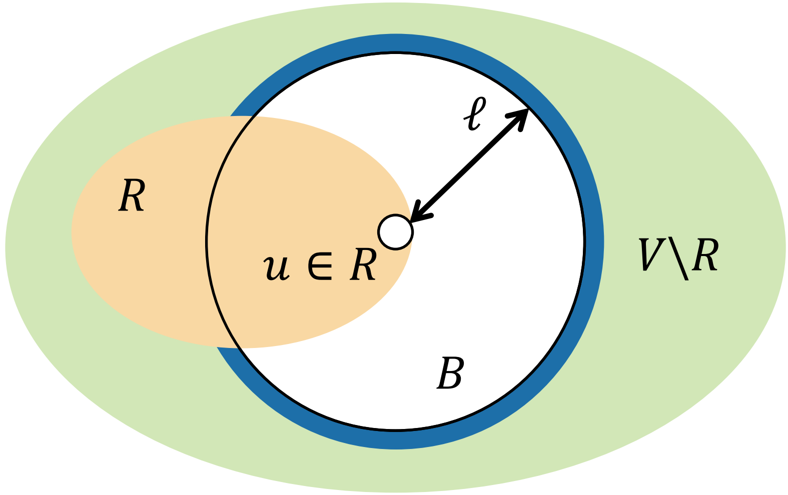
- Compute the minimum probability

$$\mu_{\min} = \min\left\{ \mu_u(X_u \mid \tau) \mid \tau \in [q]^{\partial B} : \tau_{\partial B \cap R} = X_{\partial B \cap R} \right\}$$

- Flip a coin such that (Bayes Filter)

$$\Pr[\mathrm{HEADS}] = \frac{\mu_{\min}}{\mu_u(X_u \mid \partial B \leftarrow X_{\partial B})}$$





# Unified Bayes filter

- Pick the vertex $u \in R$ with the lowest index
- Define the regime

$$B = B_\ell(u) \backslash R \cup \{u\}$$

- Compute the minimum probability

$$\mu_{\min} = \min\{ \mu_u(X_u \mid \tau) \mid \tau \in [q]^{\partial B} : \tau_{\partial B \cap R} = X_{\partial B \cap R} \}$$

- Flip a coin such that (Bayes Filter)

$$\Pr[\mathrm{HEADS}] = \frac{\mu_{\min}}{\mu_u(X_u \mid \partial B \leftarrow X_{\partial B})}$$

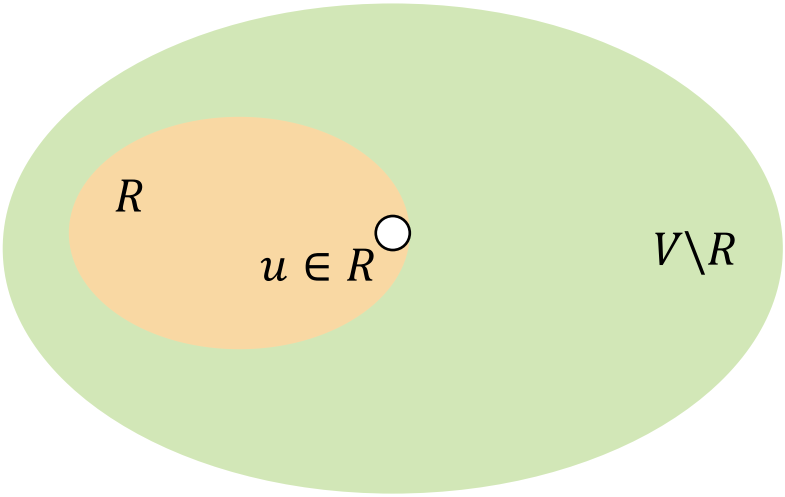
- **If** the outcome is HEADS, **then**

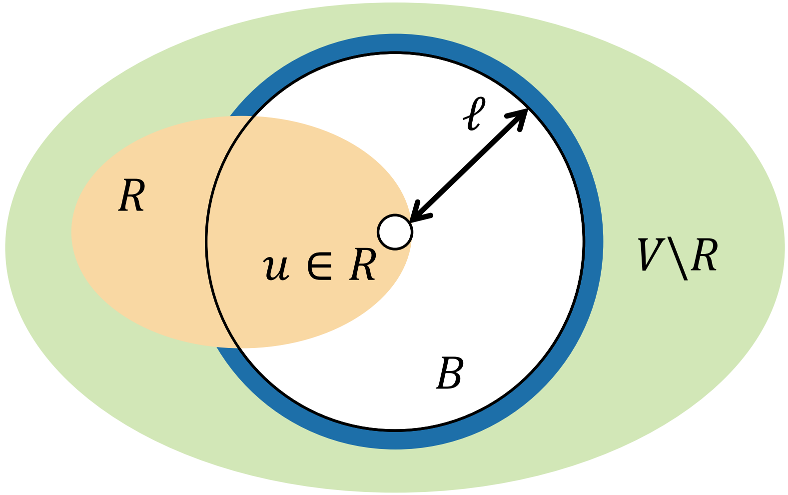
  Resample $X_B \sim \mu_B(\cdot \mid \partial B \leftarrow X_{\partial B})$

  **Return** $(X, R \backslash \{u\})$

- **If** the outcome is not HEADS, **then**

  **Return** $(X, R \cup \partial B)$

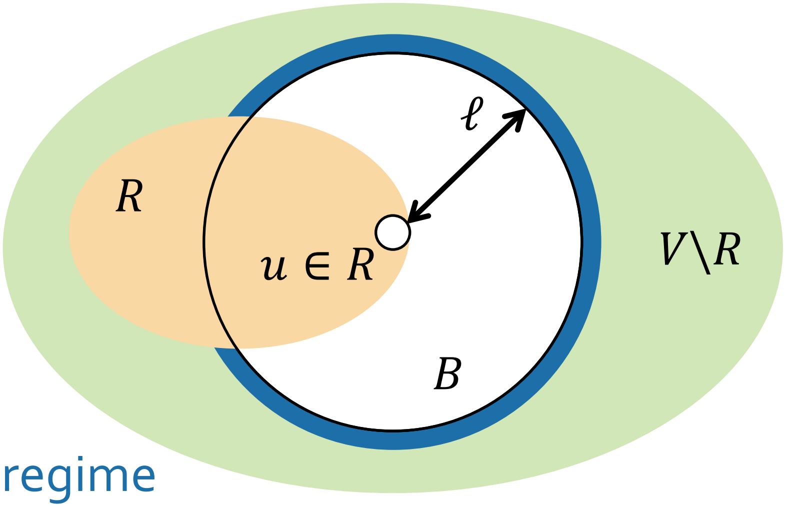




# Bayes Filter

$$\Pr[\text{HEADS}] = \frac{\mu_{\min}}{\mu_u(X_u \mid X_{\partial B})} = \frac{\mu_u(X_u \mid \partial B \leftarrow \tau^*)}{\mu_u(X_u \mid \partial B \leftarrow X_{\partial B})}$$

$$\mu_{\min} = \min\left\{ \mu_u(X_u \mid \tau) \mid \tau \in [q]^{\partial B} : \tau_{\partial B \cap R} = X_{\partial B \cap R} \right\}$$



$\tau^*$ and $X_{\partial B}$ disagree only at blue regime

**SSM** →

$$\Pr[\text{HEADS}] = \frac{\mu_u(X_u \mid \partial B \leftarrow \tau^*)}{\mu_u(X_u \mid \partial B \leftarrow X_{\partial B})} = 1 - \exp(-\Omega(\ell))$$

Pass the Bayes filter → Eliminate vertex $u$ from $R$

Not pass the Bayes filter → Add $\partial B$ into $R$, where $|\partial B| \le |S_\ell(v)| = \exp(o(\ell))$

$$\begin{aligned} \mathbb{E}[\,|\text{new } R| - |R| \mid R\,] &= -\Pr[\text{HEADS}] + (1 - \Pr[\text{HEADS}]) \cdot |\partial B| \\ &= -1 + \exp\big(-\Omega(\ell)\big) + \exp\big(-\Omega(\ell) + o(\ell)\big) \\ (\ell = \Theta(1)) \qquad &< 0 \end{aligned}$$